Combinatorics

- Fundamental principles, examples.
- Three main principles.
  (Induction, Inclusion/Exclusion, pigeonhole)
- Combinatorial proofs, recurrence relations.
- Discrete probability

Combinatorics. F1

What is combinatorics?

- the mathematics of counting.
- purely combinatorial or with combinatorial aspects.

Why we study it?

- applications in other areas of mathematics
- applications elsewhere.
  - graphs, networks
  - algorithms
  - biology ...
- easy to explain, "smart" solutions.

Some recaps on sets:

or "distinguishable"

1) The elements of a set are distinct. A set contains one of each element

2) "Union" of sets A and B: $A \cup B$, the smallest set that contains elements in A or B:

$$\{1,2\} \cup \{1,3\} = \{1,2,3\}$$

3). "Intersection" of A and B: $A \cap B$: contains only elements in common: $\{1,2\} \cap \{1,3\} = \{1\}$

$\{1,2\} \cap \{3,4\} = \phi$ ← empty set.

disjoint

4) $|A|$: cardinality, number of elements in A.

5). "partition": collection of disjoint subsets whose union is the set itself: $\{1,2,3,4\}$.

$\{1\}, \{2\}, \{3,4\}$.

How to count?

$\boxed{\text{1st law of counting}}$ Add up exclusive options.

when allowed to choose items from sets A and B, there are $|A| + |B|$ options.

e.g. when choosing to watch 3 movies or to read 4 books there are 7 options in total.

$\boxed{\text{2nd law of counting}}$: Multiply successive options

when you first choose an element from A, then choose another element from B, there are $|A| \cdot |B|$ options

Remark. 1) A and B can be different, the same, or overlaping.

2). generalises to n successive choices.

3). The "outcome" is an ordered list of successive choices.

(The order is important when A, B are not disjoint).

The order can either make a difference or not.

e.g. a) 3 shirts, 4 pants, 2 shoes, how many outfits?

$3 \times 4 \times 2 = 24$.

b) 10 coin flips, how many possibilities?

2^{10}

c) How many possible 6-digit numbers?

9×10^5 (first digit can't be 0).

d) How many words with strictly less than 5 letters?

$26^4 + 26^3 + 26^2 + 26$.

Remark. Some choices from A might result in a different $|B|$:

e.g. a) There are 12 books in total, how many ways to take 4 books out and put them on a shelf?

$12 \times 11 \times 10 \times 9$. (Order matters)

b) How many possible 6-digit distinct numbers?

$9 \times 9 \times 8 \times 7 \times 6 \times 5$

3rd law of counting

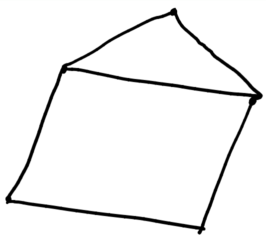
"Shepherds' law".

- When counting # of sheeps, one can count # of legs and divide by 4 (provided each sheep has 4 legs.). Similarly, when we overcount by a uniform multiplicative factor, we can correct it by dividing this factor.

when to apply?

- multiple outcomes that share a common feature.
- this outcomes are regarded as "equivalent classes"

e.g.



- We can count # of edges by summing up the degrees of each vertex
- each edge connects 2 vertices.
- $(2+3+2+3+2)/2 = 6$.

Check two things

- make sure we are overcounting uniformly.

  (each sheep has same amount of legs.)

- figure out the size of overcounting

  (the size of the legs is 4)

Remark. The shepards law can be generalised so long as the average size (# of legs) remains the same; e.g. 1 4-leg, 1 3-leg and 1 5-leg sheep.

e.g. a) a) There are 12 books in total, how many ways to take 4 books out and put them on a bag?

$$(12 \times 11 \times 10 \times 9) / (\underbrace{4 \times 3 \times 2 \times 1}).$$

(each sheep is a set of 4 books).

For any 4 books, there are $4 \times 3 \times 2 \times 1$ ways we could have taken them.

(order does not matter)

b) Out of 52 cards, how many different 5-card poker hands?

$$52 \times 51 \times 50 \times 49 \times 48 / 5 \times 4 \times 3 \times 2 \times 1.$$

$= 2{,}598{,}960$

Definition 1.1 An ordered one (string) is called a permutation of length k from set A, if every element is in A and all elements are different.

Definition 1.2. For $n = 1, 2, \dots,$ define $n! = n \cdot (n-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1$.

define $0! = 1$.

For $n \geq k$, define $P(n,k) = \dfrac{n!}{(n-k)!}$

Note that $P(n,n) = n!$

proposition 1.3 If $|A| = n$, $0 \leq k \leq n$, then there are $P(n,k)$ permutations of length k from $|A|$.

which previous examples are permutations?

(when the order does not matter).

Definition 1.4 For a set A, a combination of X is a subset of A.

e.g. How many combinations of size 2 from $\{a, b, c, d\}$?

$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$.

or. $4 \times 3 / 2 \times 1 = 6$.

Definition 1.5 For $0 \leq k \leq n$, define

$$C(n,k) := \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

Notation: $\binom{n}{k}$, $C(n,k)$, ${}_nC_k$, C^n_k ...

" n choose k ".

Proposition 1.6 If $|A| = n$, $0 \leq k \leq n$, there are $C(n,k)$ combinations of length k from n.

which previous examples are combinations?

(when the order does matter).

proposition 1.7 $\binom{n}{k} = \binom{n}{n-k}$.

proof 1. $\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k)!\,(n-(n-k))!}$

$$= \binom{n}{n-k} \qquad \boxed{}$$

proof 2. Let $|A| = n$. choose a combination of length k from A, $X \subseteq A$. Every time we choose X, then there is a subset $A \setminus X$ of size $n-k$. Therefore, the number of ways to choose X and to choose $A \setminus X$ are the same. By proposition 1.6.

choose X : $\binom{n}{k}$

choose $A \setminus X$: $\binom{n}{n-k}$

$$\Rightarrow \binom{n}{k} = \binom{n}{n-k}. \qquad \boxed{}$$

Remark. proof 1 is algebraic, proof 2 is combinatorial.