F2. Examples. Fundamental principles I

Recall 3 basic laws of counting:

$\boxed{\text{1st law}}$ Add up exclusive options.

$\boxed{\text{2nd law}}$ Multiply successive options

$\boxed{\text{3rd law}}$ "Shepherd's law"

Permutation: $P(n,k) = \dfrac{n!}{(n-k)!}$

Combination: $C(n,k) = \dfrac{P(n,k)}{k!} = \dfrac{n!}{(n-k)!\,k!}$

Some exercises.

E.X. 2.1 How many different ways to order letters A, A, B?

Sol 1. AAB, ABA, BAA.

Sol 2. $3!$ ways to permute A, A, B.

size of equivalent class: $2!$

$3!/2! = 3.$

Generalization "Mississippi rule", to count the number of distinct permutations of a string: ABCCC

- Count # of permutations as if the letters were distinct

  $A\,B\,C_1\,C_2\,C_3$

- Divide by the # of equivalent permutations.

e.g.

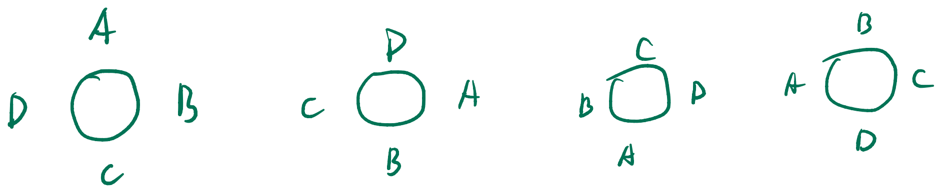
a). HAPPY.: $5!/2!$ distinct permutations.

b). NOON: $4!/(2!\cdot 2!)$

c) Mississippi: $11!/(4!\times 4!\times 2!)$.

E.X. 2.2 How many ways can n people be seated at a round table?

In this case.



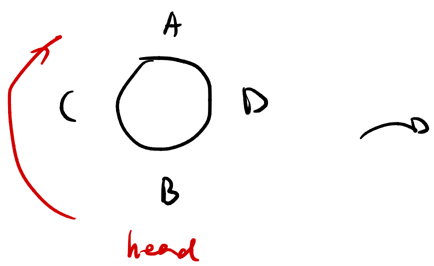
Count as the same seating.

sol. Let m be the # of ways of seating n people at a round table, and let h be the # of ways of choosing the "head" of a table.

- $h = n$.

Let us first choose a head, then read the string

clockwise :



"BCAD".

This gives all permutations. By the second counting rule (multiplication) :

(Sanity check: 3 ppl).

$$m \times h = P(n,n)$$

$$\Rightarrow m = \frac{P(n,n)}{h} = \frac{n!}{n} = (n-1)!$$

E.X.2.3 How many ways can 8 people sit at a round table if Alice and Bob must sit next to each other?

Sol 1. Treat Alice and Bob as a single unit, then seat 7 "people" around the table, there are $6!$ ways, arrange Alice and Bob gives $2 \times 6!$ ways.

Sol 2. Seat Alice anywhere. (Does not build up the size)

Choose a seat for Bob: 2 (left or right)

Arrange the rest of 6 people. $6!$

In total: $2 \times 6!$

E.X.2.4 Same as E.X.2.3, but Alice and Bob can't sit together?

Sol. ("they do not sit together"

= "they are not next to each other".

Total # $= (n-1)! = 7!$

# of ways $= 7! - 2 \times 6! = 7 \times 6! - 2 \times 6! = 5 \times 6!$

Alternatively, sit Alice anywhere, the sit Bob somewhere not next to her: 5 ways. Finally, permute the 6 people left: $6!$ ways. In total,

$$5 \times 6! \text{ ways.}$$

E.x. 2.5 How many ways to arrange 4 A's and 2 B's at a round table?

Wrong sol: By the Mississippi rule, there are

$6!/(4! \times 2!) = 15$ permutations, since there are 6 positions, there must be

$15/6 = 2.5$ ways ...

Sol: Look at the distance between 2 B's, since the rest of the circle are filled by A's. Actually, the shortest distance determines the arrangements.

The shortest distance between B's can be 0, 1, 2, so there are in total 3 arrangements.

what went wrong previously?

when applying the multiplication rule, we did not check that all permutations were distinct. e.g.

ABAABA, BAABAA are the same.

Ex. 2.6 How many ways can we paint a rubic cube with 6 colors?

Sol. say we put red at the bottom.

choose top: 5 ways.

circle.

arrange the rest of 4 colors: $3!$ ways.

In total: $5 \times 3!$ ways.

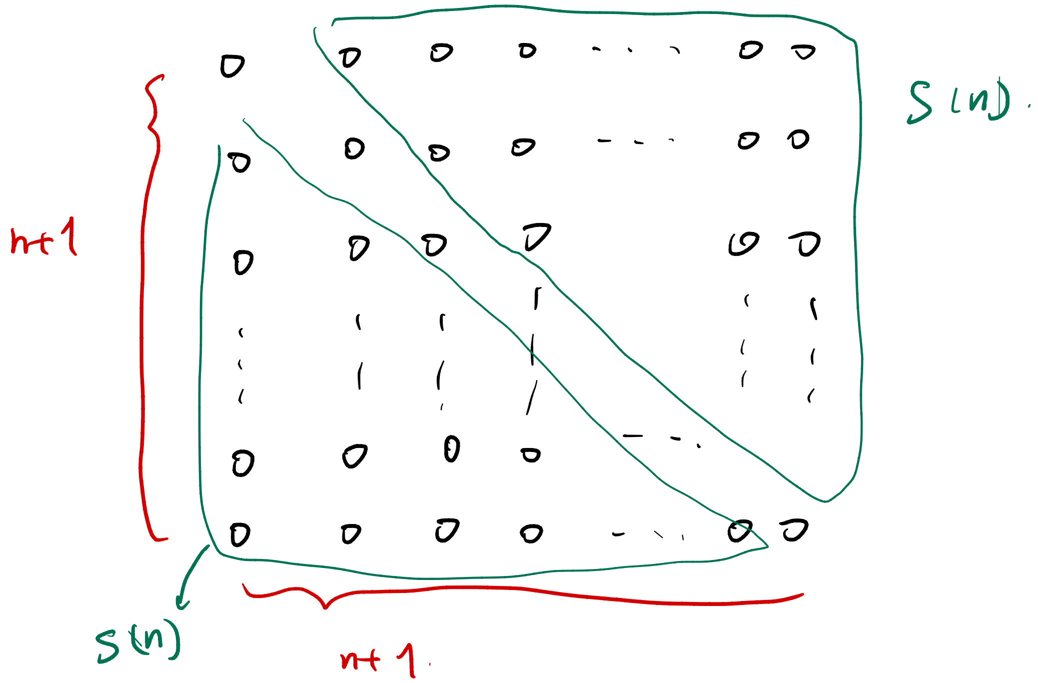
Thm 2.7 Pascal's identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

for $1 \leq k < n$.

combinatorial proof. The idea of a combinatorial proof is to give a counting argument for some identity, often by counting the same thing in two different ways.

Ex. 2.8 For $n \geq 1$, show that $S(n) := \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Look at the $(n+1) \times (n+1)$ array below, clearly, there are $(n+1)(n+1)$ points:



Count the last n points in the first column, last $n-1$ points in the second column .... we get $n + (n-1) + \cdots + 1 = S(n)$.

By symmetry, in the upper triangle, there are $S(n)$ points too.

There are $(n+1)$ points left.

$$2 \times S(n) = (n+1) \times (n+1) - (n+1)$$

$$= (n+1)n. \quad \Rightarrow \quad S(n) = \frac{n(n+1)}{2}$$ ☑

E.X.2.9: Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

We know that there are 2^n binary strings of length n.

(coin toss) $(\underbrace{0, 1, \cdots\cdots}_{n})$

Fix k between 0 and n, and count the number of strings with k 1's. Chose k positions for 1's and put 0 elsewhere, e.g. $n=7$, $k=4$: 1001101.

There are $\binom{n}{k}$ ways of choosing k positions. so there are $\binom{n}{k}$ binary strings of k 1's.

summing over all possible numbers of 1's: $(0 \cdots n)$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Prf of Pascal identity:

Algebraic prf: $\binom{n-1}{k-1} + \binom{n-1}{k}$

$$= \frac{(n-1)!}{(n-k)!(k-1)!} + \frac{(n-1)!}{k!(n-1-k)!}$$

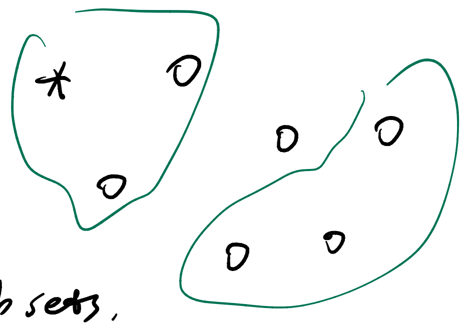
$$= \frac{k(n-1)!}{(n-k)!\,k(k-1)!} + \frac{(n-k)(n-1)!}{k!\,(n-1-k)!\,(n-k)}.$$

$$= \frac{k(n-1)!}{(n-k)!\,k!} + \frac{(n-k)(n-1)!}{k!\,(n-k)!}$$

$$= \frac{(k+n-k)(n-1)!}{k!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}.$$

Combinatorial prf Look at a set X with n objects, and label one of the objects $*$. We know that X has $\binom{n}{k}$ subsets of size k.



To count the number of subsets that includes $*$, there are $\binom{n-1}{k-1}$ ways. (remaining element)

To count the number of subsets that does not include $*$, there are $\binom{n-1}{k}$ ways.

By the first law of counting:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$