F3: Binomial thm & Induction

Last time:

Thm (Binomial theorem.)

For any real numbers x and y, for every $n \geq 0$.

$$(x+y)^n = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\text{binomial coeffecients.}} x^{n-k} y^k.$$

e.g. $(x+y)^3 = (x+y)(x+y)(x+y)$ — how many different ways to get 1 x and 2 y's ?

$$= \overset{\binom{3}{0}}{x^3} + \overset{\binom{3}{1}}{3x^2y} + \overset{\binom{3}{2}}{3xy^2} + \overset{\binom{3}{3}}{y^3}.$$

Prf. Look at $(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ terms.}}$

If we multiply everything out, we have $\{x, y\}$-strings of length n as the terms of expansion. where each term is determined by choosing x or y in each factor.

$(x+y)(x+y)\cdots\cdots(x+y) \rightarrow \{xx\cdots\cdots y\}$.

For every k, every string with k y's (and thus $(n-k)$ x's) simplifies to $x^{n-k}y^k$. There are $\binom{n}{k}$ ways to choose k y's, so the coeffecient is $\binom{n}{k}$. This holds for all k. $\square$

E.x. 3.1. Give 2 different prfs that

$$3^n = \binom{n}{0}2^0 + \binom{n}{1}2^1 + \cdots + \binom{n}{n}2^n.$$

prf 1. We count tenery strings ($\{0,1,2\}$ - strings) of length n.

- For each position there are 3 choices. so 3^n in total.

- For k between 0 and n, we count tenery strings with exactly

k 2's, and every other position has 2 possibilities. 0 or 1.

So the number of strings with exactly k 2's:

$$\binom{n}{k}\underbrace{2^{n-k}}_{\text{rest of the positions.}} = \binom{n}{n-k}2^{n-k}. =$$

summing over all k:

$$3^n = \binom{n}{\underset{\substack{\uparrow \\ n-k=0}}{0}}2^0 + \cdots + \binom{n}{n}2^n.$$

$\square$

Prf 2. By the binomial thm,

$$3^n = (1+2)^n = \sum_{k=1}^{n}\binom{n}{k}1^{n-k}2^k$$

$$= \binom{n}{0}2^0 + \cdots + \binom{n}{n}2^n.$$

$\square$

Thm 3.2 (Multinomial thm)

For any real numbers $x_1, x_2, \dots, x_r$, and any positive integer n,

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{k_1, k_2, \dots, k_r \geq 0 \\ \underbrace{k_1 + k_2 + \cdots + k_r = n}_{\text{constraints on } k_i\text{'s}}}} \binom{n}{k_1, \dots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}$$

where $\binom{n}{k_1, \dots, k_r} := \dfrac{n!}{k_1! k_2! \cdots k_r!}$ is the multinomial coefficients.

(Recall Mississippi rule), representing the number of "rearrangements".

Review of three principles.

Thm 3.3 Principle of mathematical induction

Let $S(n)$ be an open statement involving the positive integer n. If:

open statement ↳ we don't know if it's true or not

Base case : $S(1)$ is true.

Induction step : For all $k \geq 1$, if $S(k)$ is true, so is $S(k+1)$.

Then $S(n)$ is true for all $n \geq 1$.

To prove this, we need the following lemma.

Lemma 3.4. (The well-ordering principle.)

Every non-empty set of positive integers has a least element
smallest.

prf: Suppose $S(n)$ satisfies the base case and the induction step. Let set $F = \{ k \geq 1 : S(k) \text{ is false} \}$, the
collection of k's.
set of positive integers where $S(k)$ fails. Want: $F = \phi$.

Suppose F is not empty, then by Lemma 2.8, F has a least element m.

The base case holds: $1 \notin F$

m is the smallest: $m-1 \notin F$.

$\Rightarrow S(m-1)$ is true.

By induction step, so is $S(m)$. $\Rightarrow m \notin F$

This contradicts F having a least element, so F must be empty. (proof by contradiction). $\boxed{/}$

E.X. 3.5. Let $S(n)$ be: $\sum_{i=0}^{n-1} 2^i = 2^n - 1$.

Show that $S(n)$ is true for all $n \geq 1$.

prf. **Base case:** $n = 1$.

$$\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^1 - 1. \quad \checkmark$$

Induction step, Let $k \geq 1$, and assume $S(k)$ is true.

(This is called the "Induction Hypothesis").

$$\Rightarrow \sum_{i=0}^{k-1} 2^i = 2^k - 1$$

then
$$\sum_{i=0}^{k} 2^i = \sum_{i=0}^{k-1} 2^i + 2^k = 2^k - 1 + 2^k$$
$$= 2 \times 2^k - 1$$
$$= 2^{k+1} - 1.$$

Therefore, $S(k+1)$ holds.

By PMI, $S(n)$ is true for all $n \geq 1$. $\boxed{/}$

Ex. 3.6 Use the PMI and Pascal's identity to show the "hockey stick identity": for all non-negative integers

$0 \le r < n$,

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

sums of binomial coefficients.

Prf. Fix $0 \le r < n$, we perform induction on n.

Base case: $n = r$

$$\sum_{k=r}^{n} \binom{k}{r} = \sum_{k=r}^{r} \binom{r}{r} = 1 = \binom{r+1}{r+1} = \binom{n+1}{n+1}.$$

Induction step: let $m \ge r$, assume

$$\sum_{k=r}^{m} \binom{k}{r} = \binom{m+1}{r+1}$$

then for $n = m+1$:

$$\sum_{k=1}^{m+1} \binom{k}{r} = \sum_{k=1}^{m+1} \binom{k}{r} + \binom{m+1}{r}$$

$$= \binom{m+1}{r+1} + \binom{m+1}{r}$$

$$= \binom{m+2}{r+1}.$$

Pascal's identity

By PMI, this holds for all n. $\square$

Recall

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1.
\end{array}$$

$\binom{0}{0} = 1.$

1, 1, 2, 3

"Pascal triangle".

$n=4$

$(k=3)$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$\binom{7}{3}$

- Element of n th row, k th column : $\binom{n}{k}$ binomial coefficient.

- Pascal's identity.

- "Hockey stick identity".

$$\binom{7}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2}$$

$n=6$
$r=2.$

Looks like a Hockey stick.

- Fibonacci sequence. (later)

1, 1, 2, 3, 5, 8, 13, ...

Thm 3.7 (The strong Induction principle)

Let $S(n)$ be an open statement involving the positive integer n.

Let $1 \leq n_0 \leq n_1$ If :

Base case . $S(n_0), S(n_0+1), \ldots, S(n_1-1), S(n_1)$ are true, and

Induction step . For all $k \geq n_1$, if $S(n_0), S(n_0+1), \ldots, S(k-1)$, $S(k)$ are true, then so is $S(k+1)$.

then $S(n)$ is true for all $n \geq n_0$.

Prf . Assume the above conditions hold. Let $P(n)$ be the statement that " $S(n_0), S(n_0+1), \ldots, S(n_1+n-1)$ are true ".

Base case : $n=1$. $P(1)$ is true by our assumption.

Induction step : Let $k \geq 1$. assume $P(k)$ is true. Then $S(n_0), \ldots, S(n_1+k-1)$ are true. By our induction step assumption, this implies $S(n_1+k)$ is true, therefore,

$$S(n_0), \ldots, S(n_1+k-1), S(n_1+k) \text{ are true.}$$

$\Rightarrow P(k+1)$ is true.

By PMI, $P(n)$ is true for all $n \geq 1$.

Note that $P(n-n_1+1)$ implies $S(n)$. $\Rightarrow S(n)$ is true for all n.

$\boxtimes$

E.X. 3.8. You can buy mozzarella sticks in bags of 3 or 5 at MAX. Show that for any $n \geq 8$, you can buy exactly n sticks.

Prf. Let $S(n)$ be the statement.

" $n = 3a + 5b$ for some non-negative integers a, b ".

If $S(n)$ is true, then you can buy exactly 8 sticks.

Base case. $n = 8$. $n = 3(1) + 5(1)$.

$n = 9$. $n = 3(3) + 5(0)$.

$n = 10$, $n = 3(0) + 5(2)$.

So, $S(8)$, $S(9)$, $S(10)$ are true.

Induction step. Let $k \geq 10$, and assume that $S(8), S(9), \ldots, S(k)$ are true. Then $k-2 \geq 8$, so $S(k-2)$ is true.

Note that $k-2 = 3a + 5b$ for some a, b. Then

$$k + 1 = (k-2) + 3 = 3(a+1) + 5b.$$

therefore, $S(k+1)$ is true.

By Strong Induction. $S(n)$ is true for all n.

E.X. 3.9. The first few numbers in a <u>Fibonacci sequence</u> are

1, 1, 2, 3, 5, 8, 13, 21, ... More formally, the sequence is defined

<u>recursively</u> by. $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$.

Let r be the positive root of.

$$r^2 - r - 1 = 0. \quad \text{so}$$

$$r = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Show that for $n \geq 2$, $f_n \geq r^{n-2}$.

<u>prf</u>. We use Strong induction. Let $S(n)$ be the statement,

$$f_n \geq r^{n-2}.$$

<u>Base case</u>. For $n = 2, 3$.

then $f_2 = 1 \geq 1 = r^0$. ✓.

$f_3 = 1+1 = 2 \geq 1.618 = r^1$. ✓.

$S(2)$, $S(3)$ are true.

<u>Induction step</u>. Let $k \geq 3$, assume that $S(2), S(3), \dots, S(k)$

are all true, then.

$$f_{k+1} = f_k + f_{k-1}$$

$$\geq r^{k-2} + r^{k-3}$$

$$= r^{k-2}(1+r)$$

since r solves $r^2 - r - 1 = 0$.

$$= r^{k-2}(r^2)$$

$$= r^{k-1}$$

Therefore. $S(k+1)$ is true. $\Rightarrow$ $S(n)$ true for all $n \geq 2$. $\boxtimes$

Remark. This value $r = \frac{1+\sqrt{5}}{2}$ is called the golden ratio. It is known that $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = r$. $(= \varphi)$

- How come? (sketch).

Golden ratio, $\varphi = \frac{a+b}{a} = \frac{a}{b}$.

$$\Rightarrow \varphi = 1 + \frac{1}{\varphi} \Rightarrow \varphi^2 - \varphi - 1 = 0.$$

In Fibonacci: Let $a = f_{n+1}$, $b = f_n$.

$$\frac{f_{n+2}}{f_{n+1}} = \frac{f_n + f_{n+1}}{f_{n+1}} = \frac{a+b}{a}.$$

$$\frac{f_{n+1}}{f_n} = \frac{a}{b}.$$

equal if converges

"Golden spiral"