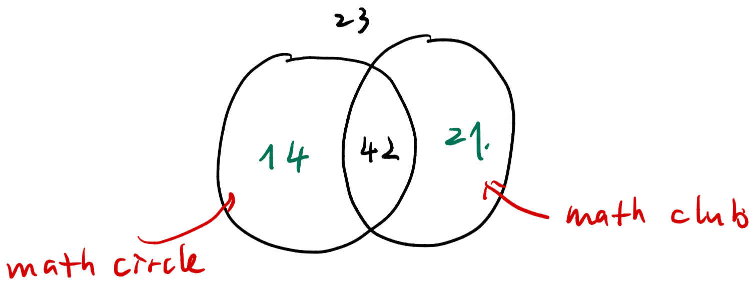
F4. Principle of Inclusion / Exclusion

E.X.4.1. Out of 100 students, 56 are registered with the math circle. 63 are registered with math club, and 42 are registered with both. How many students are registered with nothing?

(If we simply substract 63 and 56 from 100, the we get -19, which makes no sense. why?

Because we substracted the 42 students who registered in both courses twice.)

Sol: $100-63-56+42=23$.



E.X.4.2 Let X be a set, $|X| = 40$, Let $A, B, C \subseteq X$ with

$|A| = 15$, $|B| = 12$, $|C| = 14$.

$|A\cap B| = 9$, $|A\cap C| = 7$. $|B\cap C| = 4$. $|A\cap B\cap C| = 3$.

How many elements in $S \setminus (A\cup B\cup C)$?

Sol. First, $40-15-12-14$

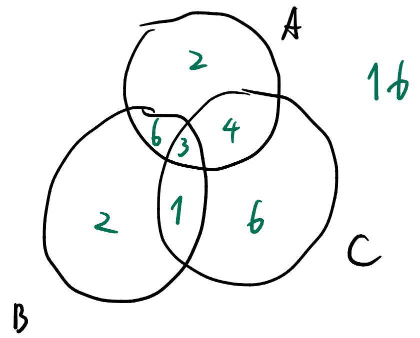
(Substracting $A\cap B$, $A\cap C$, $B\cap C$ twice.)

Second, $+9+7+4$ (add them back)

($A\cap B\cap C$ has been removed 3 times, and added back 3 times)

lastly, -3 (remove $A\cap B\cap C$).

$$|S\setminus(A\cup B\cup C)| = 40-15-12-14+9+7+4-3 = 16.$$



<u>Notation</u>. Let X be a set,

— Let $P=\{P_1, P_2, \dots, P_m\}$ be a family of properties.

e.g. P_1 = " belongs to set A ".

P_2 = " greater than 3 and smaller than 7 ".

— For $S\subseteq\{1,2,\dots,m\}$, Let $N(S)$ be the number of elements of S satisfying <u>at least</u> the properties $\{P_i, i\in S\}$:

— For $S=\phi$. $N(\phi)=|X|$. (satisfying at least nothing)

— Clearly, $N(\{1\}) \geq N(\{1,2\})$.

<u>e.g.</u> In <u>E.X.4.2</u>, Let P_1 = " belong to A "

P_2 = " — " — B "

P_3 = " — " — C ".

Then $N(\phi) = 40$, $N(\{1\}) = 15$. $N(\{2\}) = 12$, $N(\{3\}) = 14$.

$N(\{1,2\}) = 9$. $N(\{1,3\}) = 7$. $N(\{2,3\}) = 4$. $N(\{1,2,3\}) = 3$.

The # of elements satisfying none of P_1, P_2, P_3:

$$\underbrace{N(\phi) - N(\{1\}) - N(\{2\}) - N(\{3\})}_{(-1)^1} + \underbrace{N(\{1,2\}) + N(\{1,3\}) + N(\{2,3\})}_{(-1)^2} \underbrace{- N(\{1,2,3\})}_{(-1)^3}$$

Thm 4.3 (Principle of Inclusion \ Exclusion). Let X be a set and let $\mathcal{P} = \{P_1, \dots, P_m\}$ be a family of properties. The number of elements that satisfy none of the properties in $\mathcal{P}$ is given by:

$$\sum_{S \subseteq \{1,2,\dots,m\}} (-1)^{|S|} N(S).$$

E.X. 4.4 How many integer solutions are there to

$x_1 + x_2 + x_3 = 20$, with $0 \le x_1 \le 8$, $0 \le x_2 \le 10$, $0 \le x_3 \le 12$?

Sol. " $x_1 + x_2 + x_3 = 20$ "

For $x_1, x_2, x_3 \geq 0$

" Stars and bars " method:

Imagine we distribute 20 balls to 3 people:

How many ways? $\binom{20+(3-1)}{3-1} = \binom{22}{2}$.

(Remark: How many solutions to

$$x_1 + x_2 + \cdots + x_k \le n. \quad ?$$

It's equivalent to " $x_1 + \cdots + x_{k+1} = n$ ".

$$\binom{n+(k+1-1)}{k+1-1}.$$

)

Let $X = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 20,\ x_1, x_2, x_3 \geq 0\}$. then we know there are $\binom{20+3-1}{3-1} = \binom{22}{2}$ integer solutions, with $x_1, x_2, x_3 \geq 0$.

so $|X| = \binom{22}{2}$.

Let $P_1 =$ " $x_1 \geq 9$ "

$P_2 =$ " $x_2 \geq 11$ "

$P_3 =$ " $x_3 \geq 13$ ".

$$N(\emptyset) = |X| = \binom{22}{2}$$

Giving 9 1's to x_1, there are $\binom{11+3-1}{3-1} = \binom{13}{2}$ solutions when $x_1 \geq 9$, $x_2, x_3 \geq 0$. so $N(\{1\}) = \binom{13}{2}$.

Similarly, giving 11 1's to X_2, there are $\binom{9+3-1}{3-1} = \binom{11}{2}$

solutions to when $X_2 \geq 11$, $X_1, X_3 \geq 0$. So $N(\{3\}) = \binom{11}{2}$

Similarly. $N(\{3\}) = \binom{7+3-1}{3-1} = \binom{9}{2}$.

Giving 9 1's to X_1, <u>and</u> 11 1's to X_2, there are $\binom{0+3-1}{3-1} = 1$

Solution when $X_1 \geq 9$. $X_2 \geq 11$. $X_3 \geq 0$. So $N(\{1,2\}) = 1$.

There are 0 solution when $X_1 \geq 9$. $X_2 \geq 0$. $X_3 \geq 13$. $\Rightarrow N(\{1,3\}) = 0$

—— " —— $X_1 \geq 0$. $X_2 \geq 11$. $X_3 \geq 13$. $N(\{2,3\}) = 0$

—— " —— $X_1 \geq 9$. $X_2 \geq 11$. $X_3 \geq 13$. $N(\{1,2,3\}) = 0$

By thm 4.3. the number of solutions to

$X_1 + X_2 + X_3 = 0$. $0 \leq X_1 \leq 8$, $0 \leq X_2 \leq 10$, $0 \leq X_3 \leq 12$ is given by

$$N(\phi) - N(\{1\}) - N(\{2\}) - N(\{3\}) + N(\{1,3\}) + N(\{2,3\}) + N(\{1,2\}) - N(\{1,2,3\})$$

$$= \binom{22}{2} - \binom{13}{2} - \binom{11}{2} - \binom{9}{2} + 1$$

<u>Exercise</u> In how many different ways can "UPPSALA" be rearranged s.t. there are no occurances of "LAP", "UP", "SAP", or "PAL"?

<u>Sol</u> 676.

Exercise How many integers between 1 and 100 are not divisible by 2, 3 and 5?

(Sol: 26).

Application 1 Enumerating surjections:

Let A, B be two sets, and $f: A \to B$ a function.

- $f(A) = \{ b \in B : b = f(a) \text{ for some } a \in A \}$, the image of A under f.

- f is a surjection if $f(A) = B$.

- If f a surjection then $|A| \geq |B|$

Question: What's the total number of surjections from A to B?

Let $A = \{ a_1, a_2, \dots, a_n \}$,

$B = \{ b_1, b_2, \dots, b_m \}$. with $n \geq m$.

We say that a function $A \to B$ satisfies "P_i" if $b_i \notin f(A)$.

Lemma 4.5 Let $S \subseteq \{ 1, 2, \dots, m \}$, with $|S| = k$, the number of functions satisfying P_i for every $i \in S$ is $(m-k)^n$.

Prf. Denote $C = \{ b_i : i \in S \}$, then $|C| = k$. we can think of functions from A to B as strings of length n taking elements from B where f maps to.

e.g. $\{b_1, b_3, b_1, b_4, b_3\} \Leftrightarrow$ $f(a_1) = b_1$, $f(a_2) = b_3$, $f(a_3) = b_1$, $f(a_4) = b_4$, $f(a_5) = b_3$.

Then functions satisfying P_i: String of length n taking elements from $B \setminus C$. In total: $(m-k)$ elements, so there are

$(m-k)^n$ such strings. $\boxed{Q}$

Definition 4.6 For $n \geq m \geq 1$, the <u>stirling number of the second kind</u> is given by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

Thm 4.7 Let $|A| = n$, $|B| = m$, $n \geq m$. then the number of surjections from A to B is

$$S(n,m) = m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

Pf Note that a surjection is a function that fails all $P_1, P_2, \dots, P_m$.

For each k, there are $\binom{m}{k}$ subsets $S \subseteq \{1, 2, \dots, m\}$, with $|S| = k$.

For each such S, there are $(m-k)^n$ functions satisfy P_i for all $i \in S$.

Therefore by the Principle of Inclusion / Exclusion: there are

$$S(n,m) = \sum_{S \subseteq \{1,2,\dots,m\}} (-1)^{s} N(S).$$

$$= \sum_{k=0}^{m} (-1)^k \underbrace{\binom{m}{k}}_{\substack{\#\text{ of subsets} \\ \text{of size } k.}} \underbrace{(m-k)}_{N(S) \text{ when } |S| = k,}{}^{n}.$$

surjections. $\boxtimes$

E.X. 4.8 Grandma knitted 5 distinct sweaters. In how many ways can she give them to 3 grandchildren s.t. each child gets at least 1?

Sol: This is the number of surjections from {sweaters} to {grandchildren}, which is

$$S(5,3) = \sum_{k=0}^{3} (-1)^k \binom{3}{k} (3-k)^5.$$

$$= \binom{3}{0} 3^5 - \binom{3}{1} 2^5 + \binom{3}{2} 1^5 - \binom{3}{3} 0^5$$

$$= 150$$

Application 2 Derangements.

A derangement is a permutation σ s.t. $\sigma(i) \neq i$ for all i.

e.g. $\theta = 4312$ is a derangement of 1234,

$\theta = 4\,2\,1\,3$ is not.

Thm 4.9 The number of derangement d_n of $\{1, 2, \dots, n\}$ is given by :

$$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

Prf. Let P_i be the property $\theta(i) = i$. Let $S \subseteq \{1, 2, \cdots, n\}$,

the number of permutations satisfying P_i for $i \in S$ is

$$N(S) = (n - |S|)!$$

For each k, there are $\binom{n}{k}$ subsets S s.t. $|S| = k$,

So
$$d_n = \sum_{S \subseteq \{1, \dots, n\}} (-1)^{|S|} N(S)$$

$$= \sum_{k=0}^{n} (-1)^k \underbrace{\binom{n}{k}}_{\#\text{ of } S \text{ s.t. } |S|=k} \underbrace{(n-k)!}_{(n-|S|)!}$$

E.x. 4.10 You randomly send out 5 application letters to 5 universities. In how many ways can you mess up all the applications ?

Sol : $$d_5 = \sum_{k=1}^{5} (-1)^k \binom{5}{k} (5-k)!$$

$$= \binom{5}{0}5! - \binom{5}{1}4! + \binom{5}{2}3! - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5}0!$$

$$= 44.$$