F5. Pigeonhole Principle & Generating fcns

Notation: $[n] = \{1, 2, \dots, n\}$.

Pigeonhole Principle (PHP): If m objects (pigeons) occupy n places (pigeonholes) and $m > n$, then one place has at least 2 objects.

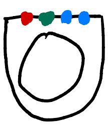
e.g. If there are 13 students in the classroom, then at least 2 have birthdays in the same month.

Generalised Pigeonhole Principle (GPHP): If m objects occupy n places, and $m > k \times n$, then at least 1 place has at least $k+1$ objects.

e.g. If there are 37 students in the classroom, then at least 4 students have birthdays in the same month.

Applications:

E.X. 5.1 A jewelery store sells rings with 4 gems placed in a row, each gem takes one of the 3 colours. Show that if the store has 82 rings, then 2 rings have identical sequence of gems.



- 82 rings are the pigeons
- each gem can be one of 3 colours, so there are

$3 \times 3 \times 3 \times 3 = 81$. sequences. so there are 81 pigeonholes.

- By PHP: 2 rings have the same sequence.

E.X. 5.2 For any $A \subseteq [200]$ with $|A| = 101$. Then there exists $m, n \in A$

s.t. $n | m$. → " n divides m ".

" m is a multiple of n ".

- elements of A be pigeons.

- For the pigeonholes, look at the 100 sets:

$$\{1, 1\times 2, 1\times 2^2, \ldots 1\times 2^i \ldots\}$$

$$\{3, 3\times 2, 3\times 2^2, \ldots 3\times 2^i \ldots\}$$

$$\{5, 5\times 2, 5\times 2^2, \ldots 5\times 2^i \ldots\}$$

$$\vdots$$

$$\{199, 199\times 2, 199\times 2^2, \ldots 199\times 2^i \ldots\}.$$

For all numbers $n \in [200]$, $n = 2^k \cdot q$ where q is an odd number.

then n goes in the pigeonhole $\{q, q\times 2, q\times 2^2 \ldots\}$.

So all 101 pigeons are in the pigeonholes, by PHP, 2 numbers

$n = q \cdot 2^{k_1}$, $m = q \times 2^{k_2}$ are in the same pigeonhole, where $k_2 > k_1$. Then

$n | m$, since $\frac{m}{n} = 2^{k_2 - k_1}$ is a whole integer.

E.X. 5.3 Take any subset $A \subseteq [9]$ with $|A| = 6$, then

A contains two elements $x, y \in A$ such that $x + y = 10$.

— The 6 elements of A are the pigeons

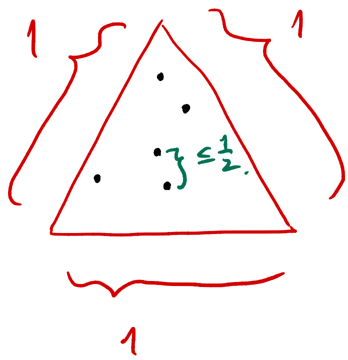
— Consider the following pigeonholes:

$\{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{5\}$.

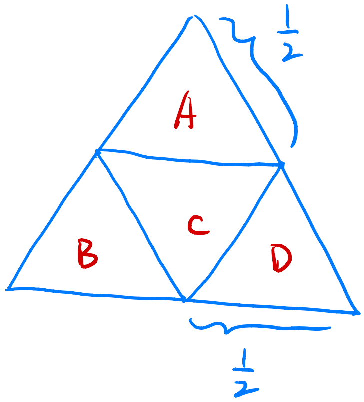
By PHP, some set contains 2 elements x and y.

Since $|\{5\}| = 1$, so the pigeonholes that contains x and y are from the other four sets. $\Rightarrow x+y = 10$.

E.x. 5.4 Suppose 5 points are placed in a equilateral triangle with side length $= 1$. Then there are two points whose distance apart is at most $\frac{1}{2}$.



Consider the 5 points to be the pigeons. Split the triangle into 4 smaller triangles A, B, C, D.



By PHP, one of A, B, C and D contains 2 points, x, y, and the maximum distance within a smaller triangle is $\frac{1}{2}$. so the distance between x and y is at most $\frac{1}{2}$.

Generating functions.

- Denote a sequence $a_0, a_1, a_2, \dots$ by $\{a_k\}_{k=0}^{\infty}$
- We associate a function

$$F(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \dots$$

- $F(x)$ is called the generating function of $\{a_k\}_{k=0}^{\infty}$.

For a fixed $n \geq 0$, consider:

E.X. 5.5. Consider the sequence $\{a_k\}_{k=0}^{\infty}$ given by $a_k = \binom{n}{k}$ for $k = 0, 1, \dots, n$, $a_k = 0$ for $k > n$.

By the Binomial theorem, the generating fcn is

$$F(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \binom{n}{x} 1^{n-k} x^k = (1+x)^n.$$

E.X. 5.6. Consider $\{a_k\}_{k=0}^{\infty}$ given by

$$\begin{cases} a_k = 1 \text{ for } k = 0, \dots, n, \\ a_k = 0 \text{ for } k > n \end{cases}$$

The generating fcn is $F(x) = \sum_{k=0}^{\infty} a_k x^k = 1 + x + x^2 + \dots + x^n$

Note that $1 - x^{n+1} = (1 + x + \cdots + x^n) - (x + \cdots + x^{n+1})$

$$= F(x) - x F(x)$$

$$= (1-x) F(x)$$

So $F(x) = \dfrac{1 - x^{n+1}}{1 - x}$.

Exercise 1 Show that the generating function for $\{a_k\}_{k=0}^{\infty}$ where $a_k = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ is given by

$$F(x) = \frac{1}{(1-x)^n}.$$

Why use generating fcns?

- allows us to use analysis to solve some combinatorial problems
- we will look at some basic uses only, in this lecture series.

Multiplication: Let $F(x) = \sum_{k=0}^{\infty} a_k x^k$, $G(x) = \sum_{k=0}^{\infty} b_k x^k$, what is the kth term of $H(x) = F(x) G(x)$?

Note that the $(k+1)$th term $= h_k x^k$.

Collecting all the x^k terms: Take $a_j x^j$: $a_j x^j b_{k-j} x^{k-j} = a_j b_{k-j} x^k$.

So the kth term in $H(x)$: $\sum_{j=0}^{k} a_j b_{k-j} x^k$.

Ex. 5.7 Use generating functions, find the number of solutions to

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = k. \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

Sol. Let $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$, $\{c_k\}_{k=0}^{\infty}$, $\{d_k\}_{k=0}^{\infty}$, $\{e_k\}_{k=0}^{\infty}$ be

the number of solutions to

$$\begin{cases} x_1 = k, \\ x_1 \geq 0 \end{cases} \begin{cases} x_2 = k, \\ x_2 \geq 0 \end{cases} \begin{cases} x_3 = k, \\ x_3 \geq 0 \end{cases} \begin{cases} x_4 = k, \\ x_4 \geq 0 \end{cases} \begin{cases} x_5 = k \\ x_5 \geq 0 \end{cases}$$

respectively, clearly, $\underbrace{a_k = b_k = c_k = d_k = e_k}_{\text{each element}} = 1$ for all $k \geq 0$. Now

Let $A(x) = B(x) = C(x) = D(x) = E(x)$ be the generating function, then

$$A(x) = \sum_{k=0}^{\infty} x^k$$

Note that $1 = (1 + x + \cdots + x^k + \cdots) - (x + x^2 + \cdots + x^{k+1} + \cdots)$

$$= (1 - x) A(x)$$

$$\Rightarrow A(x) = \frac{1}{1-x}.$$

Now let $S(x) = A(x) B(x) C(x) D(x) E(x) = \frac{1}{(1-x)^5}$, then the

kth coefficient of $S(x)$ is the sum over all solutions to

$$x_1 = k_1, x_2 = k_2, \ldots, x_5 = k_5.$$

$$\text{s.t. } k_1 + k_2 + k_3 + k_4 + k_5 = k.$$

Since $S(x) = \frac{1}{(1-x)^5}$, by exercise 1, the kth coefficient is

$$\binom{n+k-1}{n-1} = \binom{5+k-1}{5-1} = \binom{k+4}{4}.$$

Ex. 5.8 (Similar)

How many integer solutions are there to

$x_1 + x_2 + x_3 = k$, s.t. $0 \le x_1 \le 5$, x_2 even, x_3 is a multiple of 6 ?

Sol. Let $\{a_k\}_{k=0}^{\infty}$ be that a_k is the number of solutions to

$x_1 = k$, $0 \le x_1 \le 5$, then.

$(a_0 = a_1 = \cdots = a_5 = 1, \; a_n = 0, \; n \ge 6).$

$$F_a(x) = \sum_{k=0}^{\infty} a_k x^k = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1-x^6}{1-x}$$

Similarly. Let $\{b_k\}_{k=0}^{\infty}$ —— " ——

$x_2 = k$, x_2 even

$$F_b(x) = \sum_{k=0}^{\infty} b_k x^k = 1 + x^2 + x^4 + \cdots = \sum_{k=0}^{\infty} (x^2)^k = \frac{1}{1-x^2}.$$

Let $\{c_k\}_{k=0}^{\infty}$ —— " ——.

$x_3 = k$, x_3 is a multiple of 6.

$$F_c(x) = 1 + x^6 + x^{12} + \cdots$$

$$= \sum_{k=0}^{\infty} (x^6)^k = \frac{1}{1-x^6}$$

Let S_k be the number of solutions to the original system, then

$$S(x) = \sum_{k=0}^{\infty} s_k x^k = F_a(x) F_b(x) F_c(x)$$

$$= \frac{\cancel{1-x^6}}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{\cancel{1-x^6}}$$

$$= \frac{1}{(1-x)(1-x^2)}$$

$$= \frac{1}{(1-x)^2(1+x)}$$

Let $S(x) = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$

$$\Rightarrow A = \frac{1}{4},\ B = \frac{1}{4},\ C = \frac{1}{2}.$$

So $S(x) = \frac{1}{4}\left(\frac{1}{1-(-x)}\right) + \frac{1}{4}\left(\frac{1}{1-x}\right) + \frac{1}{2}\left(\frac{1}{(1-x)^2}\right)$

$$= \frac{1}{4}\sum_{k=0}^{\infty}(-x)^k + \frac{1}{4}\sum_{k=0}^{\infty} x^k + \frac{1}{2}\sum_{k=0}^{\infty}(k+1)x^k$$

Therefore, $s_k = \frac{(-1)^k}{4} + \frac{1}{4} + \frac{1}{2}(k+1)$

Remark. The generating fcns we have seen so far are "combination-like" sequence, since they don't grow too fast. There are also "permutation-like" sequences that grow faster, we need to alter the definition

Def 5.9. For a sequence $\{a_k\}_{k=0}^{\infty}$, define the Exponential generating function to be

$$F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$

E.X. 5.10 If $a_k = 1$, $k \geq 0$, then

$$F(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

E.x. 5.11 Recall $P(n,k) = \frac{n!}{(n-k)!}$. For a fixed n, let $a_k = P(n,k)$ for $0 \le k \le n$, and $a_k = 0$ for $k > n$. then the exponential generating fcn is

$$F(x) = \sum_{k=0}^{n} \frac{a_k}{k!} x^k = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

(gen fcn for $\binom{n}{k}$ and exp gen fcn for $P(n,k)$)

E.x. 5.11 Let $a_k = \begin{cases} 0 & k \text{ even} \\ 1 & k \text{ odd} \end{cases}$, then

$$F(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) - \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)$$

$$= \frac{1}{2}e^x - \frac{1}{2}e^{-x} = \frac{e^x - e^{-x}}{2}.$$

E.x. 5.12 $a_k = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd.} \end{cases}$

$$F(x) = \frac{e^x + e^{-x}}{2}. \quad \text{(check!)}$$