F6: Recurrence relations

We have explicitly defined a lot of types of numbers. For example, we defined:

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

$$P(n,k) = n!/(n-k)!$$

$$\binom{n}{k} = n!/k!(n-k)!$$

Sometimes explicit formula is difficult (or impossible) to find. so it's convinient to have a recursive definition:

e.g. $n!$ can be defined as

$$\begin{cases} 1! := 1 \\ n! := n(n-1)! & n \geq 2. \end{cases}$$

e.g. (Fibonacci sequence).

$$\begin{cases} F_1 = 1 \\ F_2 = 1. \\ F_n = F_{n-1} + F_{n-2}. \end{cases}$$

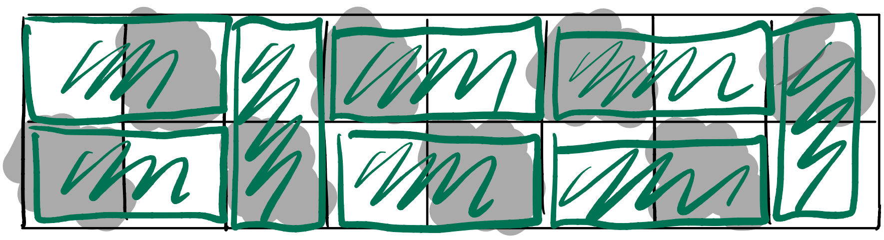
e.g. Instead of using the explicit formula for $\binom{n}{k}$, we can use pascal's identity to define:

$$\begin{cases} \binom{0}{0} = 1 \\ \binom{n}{0} = 1 \\ \binom{n}{n} = 1 \\ \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}, & n \geq r \geq 1 \end{cases}$$

Solve problems recursively

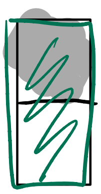
To solve some combinatorial problems, we can start by finding a recursive formula for the solution.

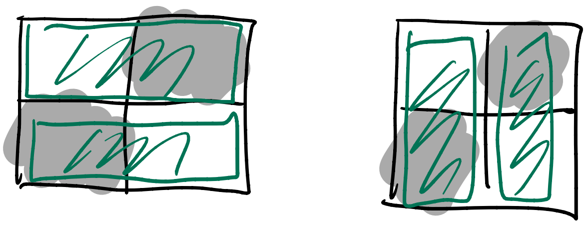
E.X.6.1 Consider a $n \times 2$ checkboard, and a set of 1×2 and 2×1 domino pieces. In how many ways can you cover the checkboard with the domino pieces?



Let f_n be the number of ways of tiling a $2 \times n$ checkboard.

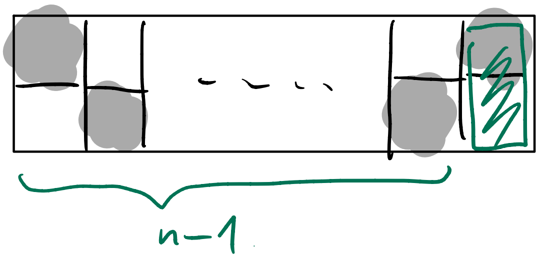
then $f_1 = 1$. $f_2 = 2$:



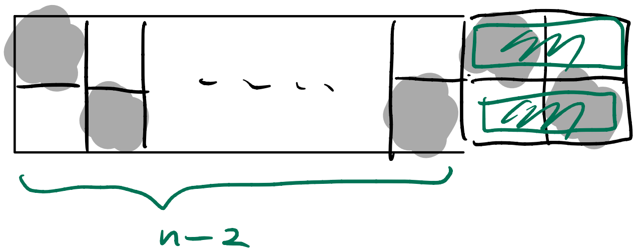


For $n \geq 3$, we consider a tile of $2 \times n$ board by either of the two following:

i) a $2 \times (n-1)$ tiling and add a verticle tile:



ii) a $2\times(n-2)$ tiling and add 2 horizontal tiles:



There is no other way. So we have:

$$\begin{cases} f_1 = 1, \quad f_2 = 2 \\ f_n = f_{n-1} + f_{n-2} \quad n \geq 3. \end{cases}$$

E.X. 6.2 How many $\overset{\text{3-ary}}{\text{ternary}}$ strings of length n do not contain 12 as a substring?

Sol. Let S_n be the number of such strings. then $S_0 = 1$ (empty string), $S_1 = 3$ $(0, 1, 2)$, $S_2 = \underbrace{3^2 - 1}_{}$ $= 8$

because it contains $\{12\}$.

For $n \geq 3$, we construct such a string by taking a "12-avoiding" string s of length $n-1$, and adding 0, 1, or 2 at the beginning to make it length n. However, if s started with a 2, we can't add 1.

we can have an 12-avoiding sequence of length $n-1$ by taking such a sequence of length $n-2$, and add a 2 on top.

So the number of sequences of length $n-1$ with 2 as its first digit is $\underbrace{S_{n-2}}_{\text{add a 2 on top of (n-2)-sequence}}$.

Therefore,

$$S_n = \underbrace{3S_{n-1}}_{\substack{\text{add 0, 1, 2 on} \\ \text{a sequence of length } n-1}} - \underbrace{S_{n-2}}_{\substack{\text{add 1 on top of} \\ \text{a sequence starting with 2.}}}$$

Therefore,
$$\begin{cases} S_0 = 1 \\ S_1 = 3 \\ S_2 = 8. \\ S_n = 3S_{n-1} - S_{n-2}, \quad n \geq 3. \end{cases}$$

Exercise 1 (Cat on stairs).

Consider a cat that jumps 1 or 2 stairs each time.

How many different ways can it jump up on the top of the stairs with length n?

Sol. Let $f(n)$ be the number of ways to reach the stair,

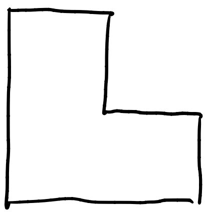
The final jump is either of length 1 or 2. therefore

$$f(n) = \underbrace{f(n-1) + f(n-2)}_{\text{why?}}$$

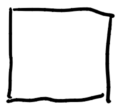
where $f(1) = 1$, and $f(2) = 2$.

(Same as the domino example !)

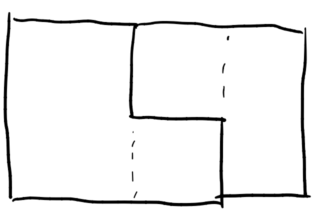
Exercise 2 Suppose you want to tile a $2 \times n$ chessboard with the following two types of tiles.



and



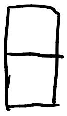
where all the rotations are allowed. e.g. a 2×3 tile.



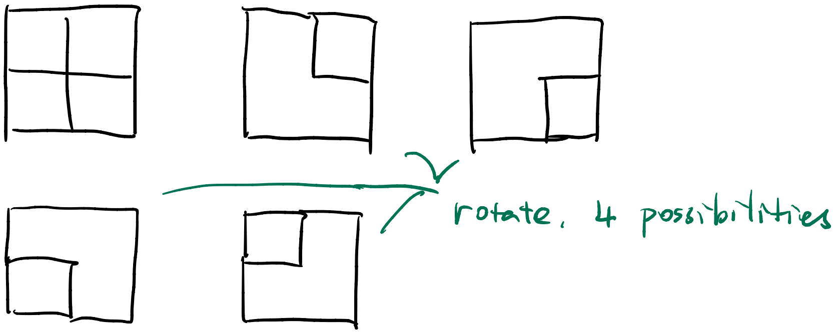
Let a_n be the number of ways of tiling a $2 \times n$ board. Find a_n.

Sol. First we will find a_1, a_2, a_3.

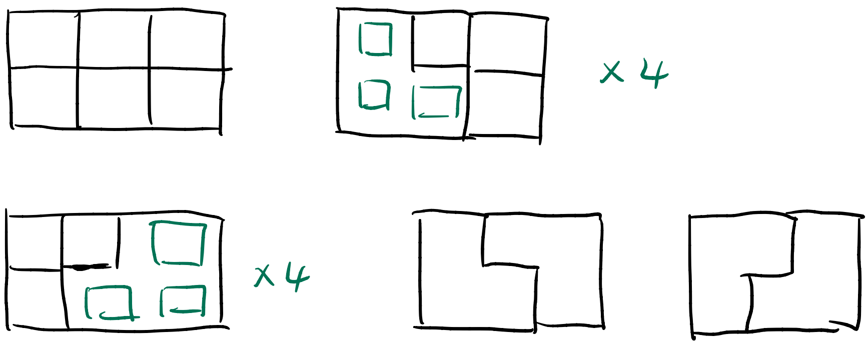
$a_1 = 1$:



$a_2 = 5$:



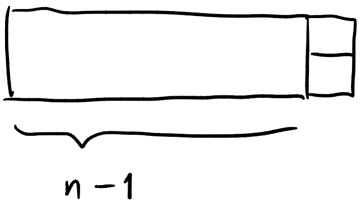
$a_3 = 11$. (area = 6)



We can form a $2 \times n$ tiling from previous $2 \times k$ tilings.

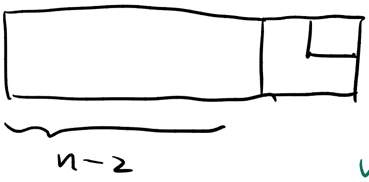
$k < n$.

— add 1 column on $2 \times (n-1)$ by

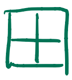


→ 1 way

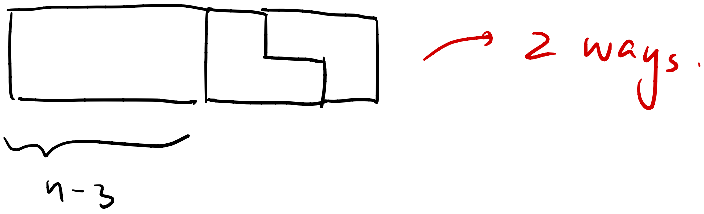
— add 2 columns on $2 \times (n-2)$ by



→ 4 ways, as in a_2

why not  ?

— add 3 columns on $2 \times (n-3)$ by



Therefore, $a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3}$, for $n \geq 4$.

E.X. 6.3 (Padovan sequence) is defined as follows:

$$\begin{cases} P_0 = P_1 = P_2 = 1 \\ P_n = P_{n-2} + P_{n-3} \end{cases}$$

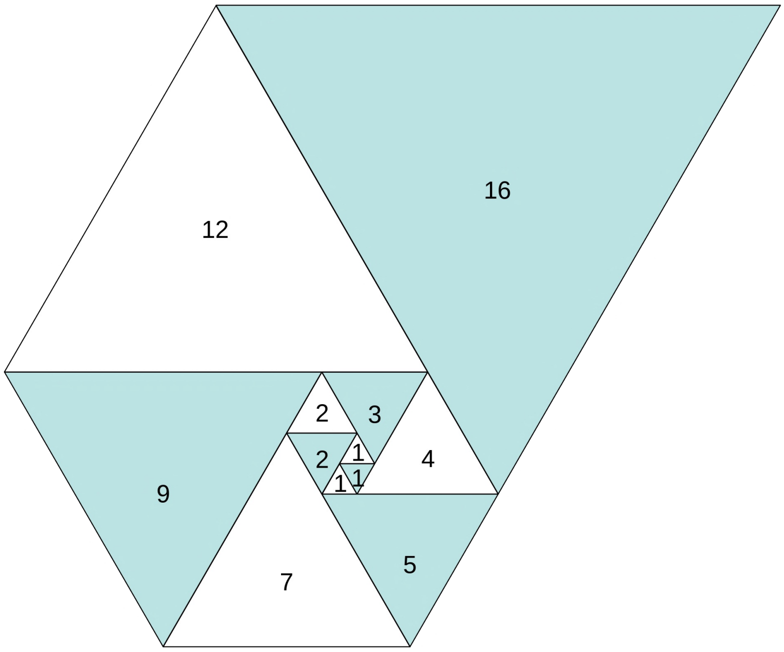
The first few values of P_n are:

1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28 ...

Recall that Spirals of squares have side lengths that follow the Fibonacci sequence.

Now spirals of equilateral triangles have side lengths that follow the Padovan sequence.

(Illustration Next Page)

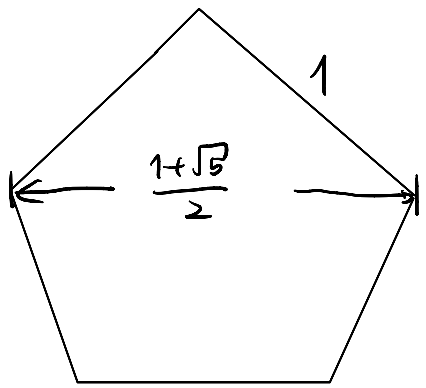


E.X. 6.4 (Pell numbers) defined as follows.

$$\begin{cases} P_0 = 0 \\ P_1 = 1 \\ P_n = 2P_{n-1} + P_{n-2}. \end{cases}$$

Recall: golden ratio: $\varphi = \frac{a}{b} = \frac{a+b}{a}. \left(= \frac{1+\sqrt{5}}{2} \right)$

pentagon:

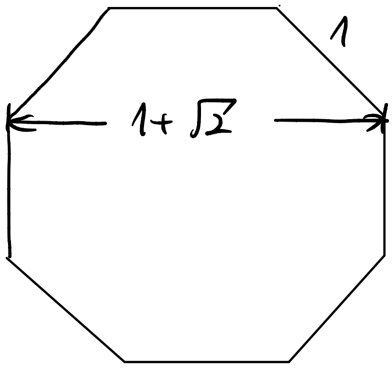


spiral - - - .

Silver ratio:

$$\frac{2a+b}{a} = \frac{a}{b} =: \delta_s = 1+\sqrt{2} = \lim_{n\to\infty} \frac{P_{n+1}}{P_n}$$

Octagon:



Ex. 6.5 (Triangular numbers)