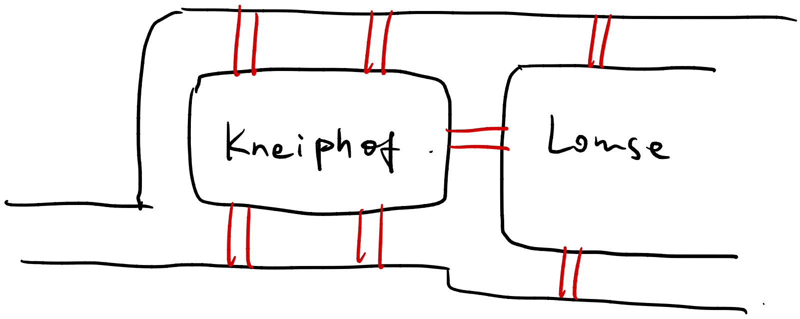
F7. Graphs.

Topic 1. Bridge of Kōnisberg.

In the 18th century, the Prussian city of Kōnisberg was divided into four different landmasses by the Pregel river. which are connected by 7 bridges.



- Two bridges each from mainland to kneiphof.
- One bridge each from mainland to lomse
- One bridge connecting kneiphof and Lomse.

Euler invested if one could give a tour through the city traversing all bridges exactly once. In 1736, he showed it's impossible to find such a route, while laying the foundations of graph theory.. Idea:

- represent landmasses by abstract points (vertices)
- represent bridges by line segments (edges)

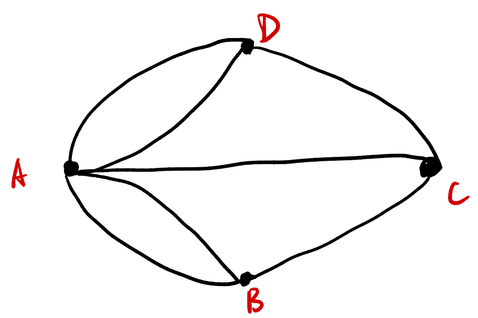
This makes what we call "multigraph" today:

Def 7.1. A multi-graph is a triple $G = (V, E, r)$ where

- V is a set of vertices
- E is a set of edges.
- $r : E \to \{(x, y) : x, y \in V\}$ assigning each edge an unordered pair of endpoint nodes.

In a word: a graph is a collection of nodes and edges.

Based on this definition, the map can be represented as.



Def 7.2. An Euler walk in an undirected graph is a walk (a sequence of alternating vertices and edges) such that each

edge is used exactly once. If such a walk exists, the graph is called traversable..

We will show that our graph is not!

Idea. Call the starting vertex S and finishing vertex V.

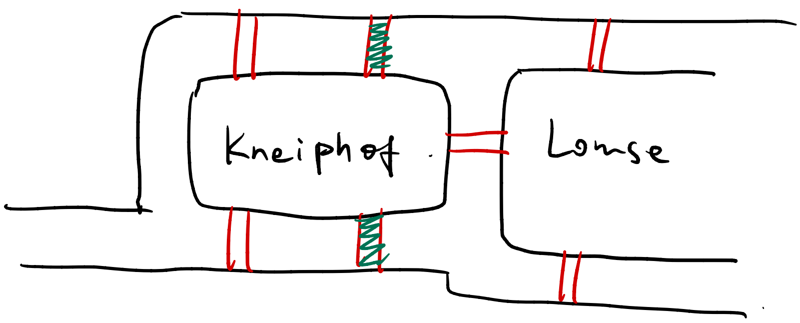
critical observation: every vertex that is not S or V must have an even degree (number of edges connected.).

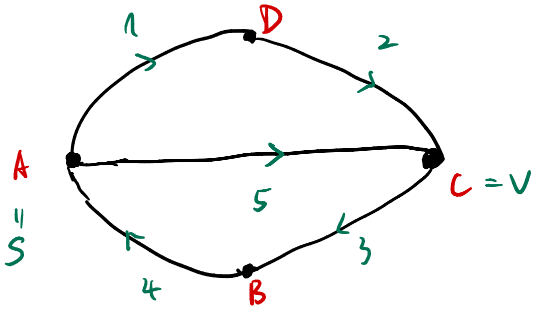
why? The only way to pass a vertex is to enter through an edge and exit through another one.

Our graph has 4 vertices (≥ 3) of odd degrees, it's not possible to find such a route.

All bridges were bombed in 1944, only 5 of them rebuilt,

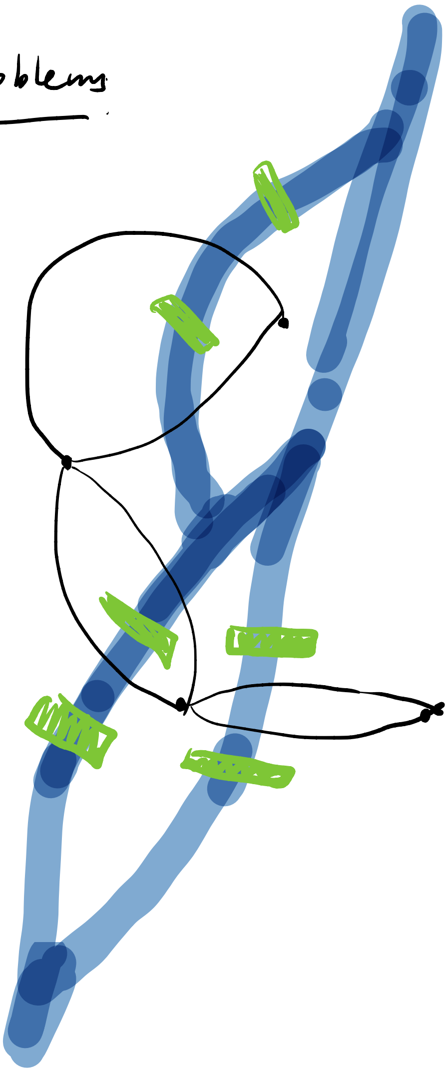
Q: Find an Euler walk in the modern day map.

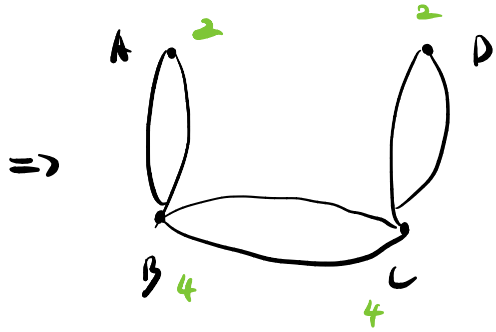




Solve the following problems

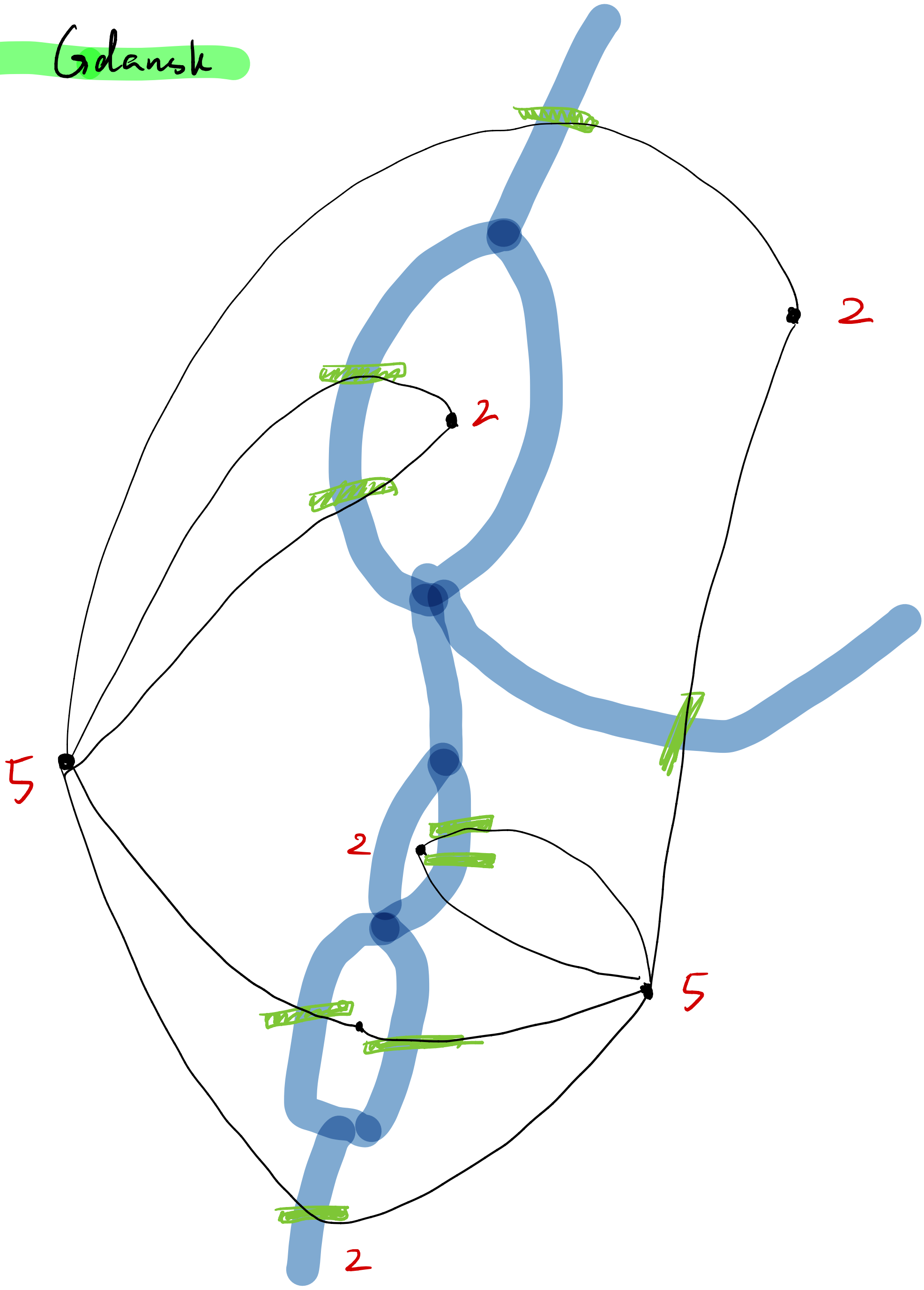
1. Budapest





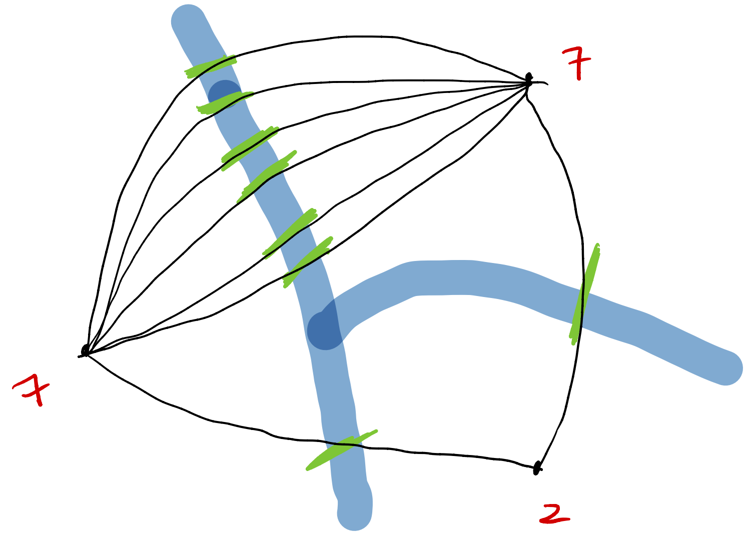
No vertex of odd degree. possible!

2. Gdansk



2 vertices with odd degrees, possible!

3. Uppsala



possible !

Not intresting ...

Topic 2 Graph coloring.

A vertex coloring is a way of assigning a color to each vertex, s.t two adjacent vertices (linked by an edge) never have the same color.

e.g. creating a map of a contry.

- It's interesting to know the minimal number of colors we need to color its vertrices. it is called the chromatic number of the graph.

E.X.7.3. Party problem.

There are 12 mathematicians celebrating their birthdays in the same week. A, B, C, ..., L. There are some people that are invited to both parties since they are friends with multiple of them :

e.g. Yu is invited to both Daniel & Elin's party

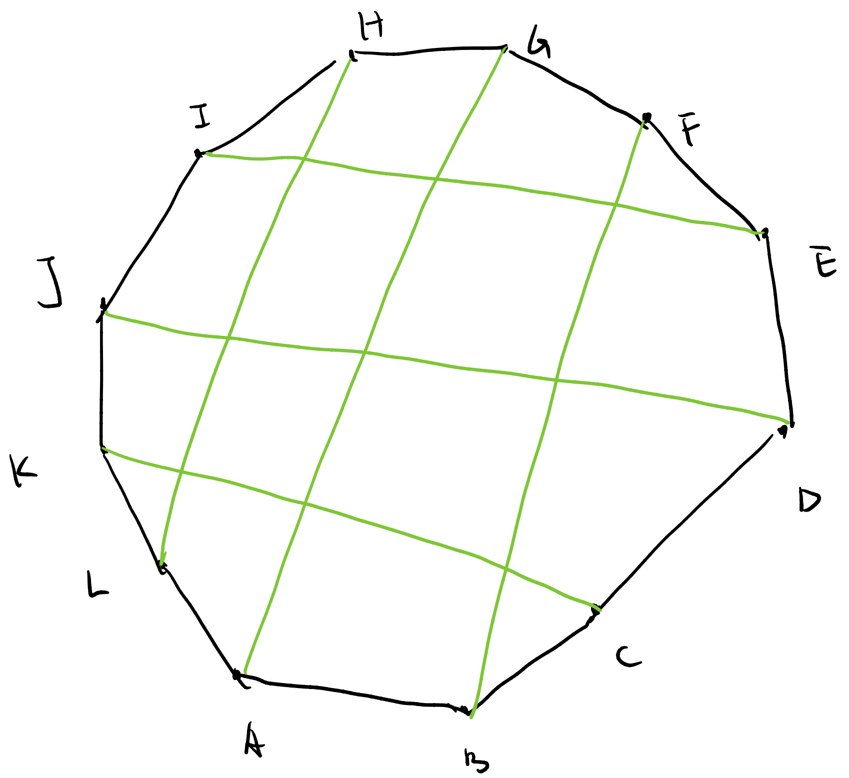
A list of people with common friends:

- A − B, A − G, A − L
- B − C, B − F.
- C − D, C − K.
- D − E, D − J.
- E − F. E − I.
- F − G.
- G − H.
- H − I, H − L.
- I − J
- J − K
- K − L

What's the minimal number of time slots for people to have parties?

⇕

minimal # of colors to color the following graph:



· Edge : common friends (cannot have the same time slots).

- Assume we can only use 2 colors: red and blue.

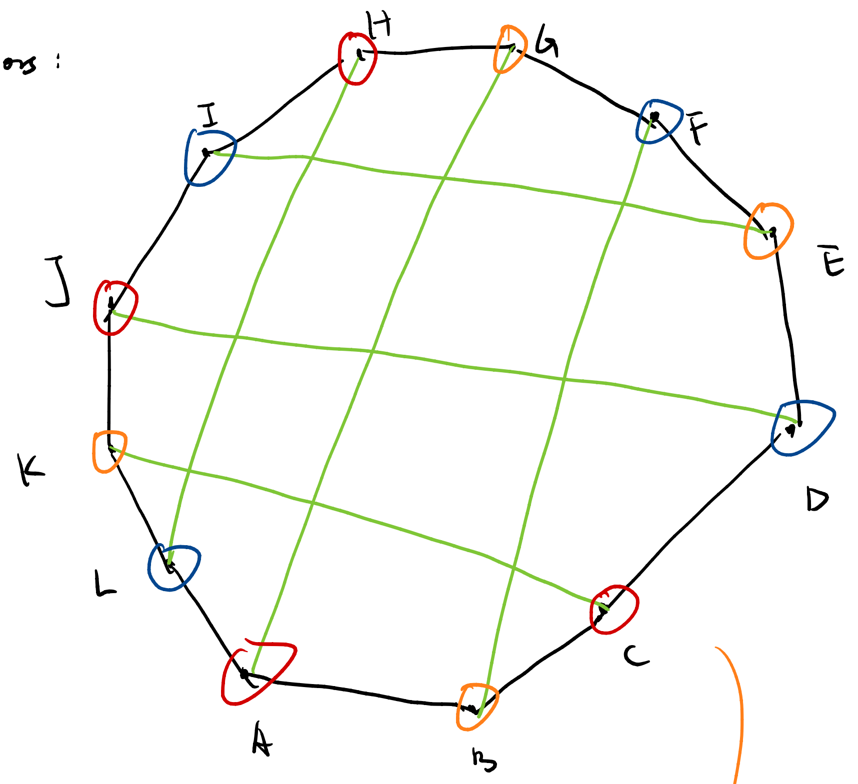
- Let A be red.

  $\Rightarrow$ L, B must be blue.

  $\Rightarrow$ K must be red.

  $\Rightarrow$ C must be blue

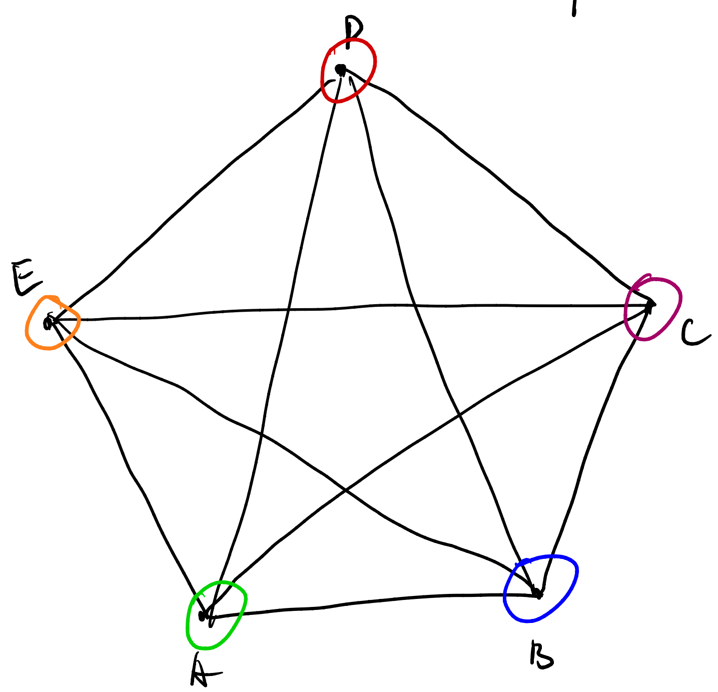
  (but B is already blue)



2 is not enough! we need another color for C.

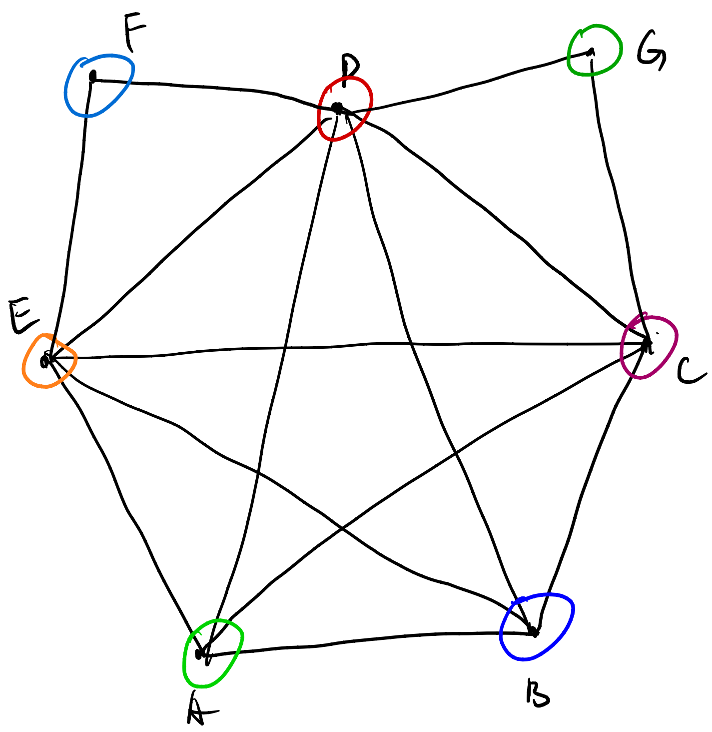
a possible coloring.

Solve the following problem: 5 students want to attend each other's parties.



we need 5 colors because the graph is fully connected. This graph is called K_5.

One step more:



· We can't use less than 5 colors.

· F and G do not require more colors. (only connected to 2).

Topic 3. Shannon switching game.

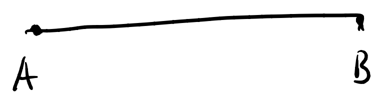
Consider a game played on a graph with two types of vertices. A, B. Each edge can be either colored or removed. There are two players. Short, and Cut, who take turns to move.

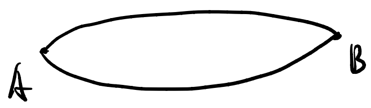
- On cut's turn, they delete a non-colored edge
- On short's turn, they color a existing edge

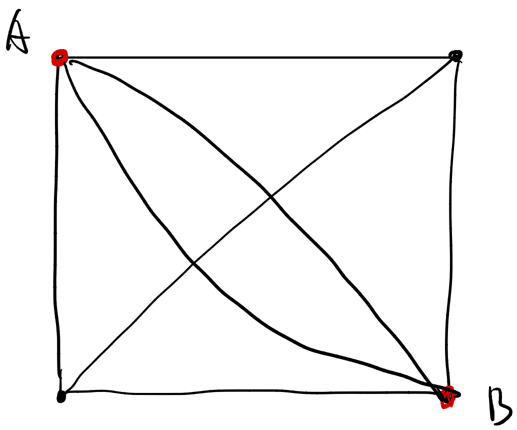
Cut wins if A and B are no longer connected

Short wins if a colored path is connected between A and B.

e.g. who will win?

i)  whoever goes first

ii)  Short

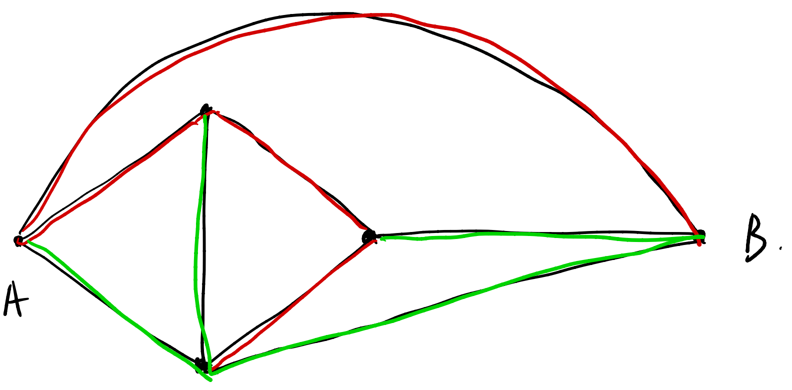
iii) 

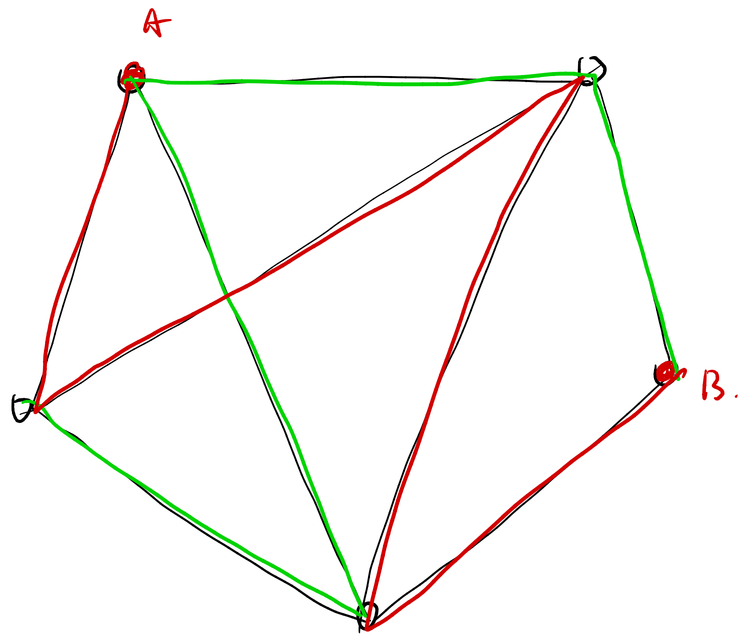
In this case, short always wins even if cut has the first turn!

- A spanning tree in a graph is a selection of edges where the edges are all connected and there are no cycles.

- Short ALWAYS wins if and only if there are sub-trees S.T. containing both A and B but don't have any common edges.

• Strategy: whenever cut deletes an edge, find an edge in the other tree to color such that it fix the broken tree.





What happens otherwise? Either Cut has the winning strategy or the "first mover" has the winning strategy.

Exercise. Choose your character and design your own graph s.t. you have the winning strategy.