Optimal Stopping with Discrete Costly Observations

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Outline

Introduction

Problem Setting

- Optimal stopping problem
- The underlying process X_t is a Markov process
- We can stop the process only at the observation points
- We can choose the sequence of time points at which we observe
- Each observation comes with a fixed cost

Our Goal

- To distribute the future observation times
- To determine when to stop
- To maximize the expected reward minus the expected cost of observations.

Introduction

Motivation and Possible Applications

- Pricing a real option based on a physical contract
- Best time for sustainable fishing
- Experimental design

Outline

Problem Formulation

- Let X be a real-valued strong Markov process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}_x)$, $X_0 = x$.
- Let $g : \mathbb{R} \to [0, \infty), g \leq M$ be a real-valued nonnegative bounded function.
- Consider a strictly increasing sequence $\gamma = (\gamma_k)_{k=0}^{\infty}$ of finite random times with $\gamma_0 = 0$, and γ_i is $\sigma\{X_{\gamma_1}, X_{\gamma_2}, \ldots, X_{\gamma_{i-1}}, \gamma_1, \gamma_2, \ldots, \gamma_{i-1}\}$-measurable.
- Denote the collection of such sequences Γ.

Problem Formulation

Definition 2.1

Define the information associated with an observation sequence $\gamma \in \Gamma$ up to time t as

$$\mathscr{G}_t^{\gamma} = \sigma\{X_{\gamma_1}, X_{\gamma_2}, \ldots, X_{\gamma_i}, \gamma_1, \gamma_2, \ldots, \gamma_i; \;\; i = \max\{j : \gamma_j \leq t\}\}.$$

Definition 2.2

Define the set of $\mathscr{G}^{\gamma}$-stopping times that take values in the set $\{\gamma_k(\omega), k \geq 0\}$ as

$$\mathscr{S}^{\gamma} = \{\tau : \mathscr{G}^{\gamma} \text{ - stopping times}, \;\; \mathbb{P}(\exists i, \tau(\omega) = \gamma_i(\omega)) = 1\}$$

The Value Function

- Let $r > 0$ be the constant discount rate,
  let $c > 0$ be the cost of observation,
- Let $V : [0, +\infty) \to [0, +\infty)$ be the value function defined by

$$V(x) = \sup_{\gamma \in \Gamma} \sup_{\tau \in \mathscr{S}^{\gamma}} \mathbb{E} \left[e^{-r\tau} g(X_{\tau}) - \sum_{i=1}^{\infty} c e^{-r\gamma_i} 1_{\{\gamma_i \leq \tau\}} \right].$$

- Clearly $g(x) \leq V(x) \leq M$.

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Our Approach

The discrete observation cost forces a discrete structure upon the problem

$\Longrightarrow$ Discrete time Markovian approach.

- Define an associated operator
- Find its fixed point
- Prove that it characterizes V
- Provide an optimal strategy in terms of the fixed point

The Discrete Structure of the Problem

Make a choice after each observation:

- Stop now;
- Make another observation.

$\Longrightarrow$ Define an operator to characterize the choices at each observation time.

Define the Operator

Definition 3.1

Denote by

$$F := \{f : [0, +\infty) \to [0, +\infty) \text{ Borel-measurable}, \quad f \geq g\}$$

the set of functions that dominate the payoff g, and define an operator $\mathscr{J} : F \to F$ by

$$(\mathscr{J} f)(x) = \max \left\{ g(x), \ \sup_{t \geq 0} \mathbb{E}_x[e^{-rt}(f(X_t) - c)] \right\}.$$

Existence of the Fixed Point

We will prove that V is a fixed point of the operator $\mathscr{J}$. Define a sequence of functions recursively by

$$\begin{cases} f_0 = g \\ f_{n+1} = \mathscr{J} f_n \quad n \geq 0. \end{cases}$$

Lemma 3.2

$\{f_n\}_{n\geq 0}$ is an increasing sequence.

Define

$$f_\infty := \lim_{n\to\infty} f_n.$$

Clearly f_∞ exists and $f_\infty \in F$.

The Fixed Point

Theorem 3.3

The function $f_\infty \in F$ is a fixed point of $\mathscr{J}$.

Proof.

$\forall n$, we have $f_\infty \geq f_{n+1} \geq f_n$. Since $\mathscr{J}$ is monotonic, we have

$$\mathscr{J} f_\infty \geq \mathscr{J} f_{n+1} \geq \mathscr{J} f_n = f_{n+1}$$

Taking limit on both sides, we have: $\mathscr{J} f_\infty \geq f_\infty$.

The Fixed Point

Proof.

(Cont.) For the other direction, fix x and let:

$$t_\infty := \inf\{t : \mathbb{E}_x[e^{-rt} f_\infty(X_t) - c)] \text{ attains its maximum}\}$$
$$\Longrightarrow f_{n+1}(x) \geq \max(g(x),\ \mathbb{E}_x[e^{-rt_\infty}(f_n(X_{t_\infty}) - c)])$$

Taking limit on both sides gives $f_\infty \geq \mathscr{J} f_\infty$. □

Uniqueness in The Wald-Bellman Equation

- Find a supermartingale
- Find a martingale
- Apply optional sampling theorem

Supermartingality

Given $\gamma \in \Gamma$, define a discrete-time filtration $\mathscr{F}^\gamma$ indexed by k by setting $\mathscr{F}_k^\gamma = \mathscr{G}_{\gamma_k}^\gamma$.

Lemma 4.1

Assume that $\hat{V} \in F$ is a fixed point of $\mathscr{J}$, and take any $\gamma \in \Gamma$. Then the process

$$Y_k := e^{-r\gamma_k}\hat{V}(X_{\gamma_k}) - c \sum_{1 \le i \le k} e^{-r\gamma_i}$$

is a supermartingale with respect to $\mathscr{F}_k^\gamma$.

The proof follows from the strong Markov property.

A Specific Sequence of Observation Times

Define the continuation set $\mathscr{C}$ and stopping set $\mathscr{D}$ as

$$\mathscr{C} := \{x \in \mathbb{R} : \hat{V}(x) > g(x)\}$$
$$\mathscr{D} := \{x \in \mathbb{R} : x \notin \mathscr{C}\}.$$

Define:

$$t(x) = \begin{cases} \inf\left\{t \geq 0 : \hat{V}(x) = \mathbb{E}_x[e^{-rt}(\hat{V}(X_t) - c)]\right\}, & x \in \mathscr{C} \\ 1, & x \in \mathscr{D} \end{cases}$$

Remark: $t(x) \in (0, \infty)$.

A Specific Sequence of Observation Times

Define $\gamma^* = (\gamma_k^*)_{k\geq 0}$ by setting $\gamma_0 = 0$ and

$$\gamma_{k+1}^* = \gamma_k^* + t(X_{\gamma_k^*}), \qquad k \geq 0.$$

Furthermore, for $\varepsilon > 0$ let

- $\tau^\varepsilon = \inf\{\gamma_k^* : \hat{V}(X_{\gamma_k^*}) \leq g(X_{\gamma_k^*}) + \varepsilon\}$,
- $\tau^* = \inf\{\gamma_k^* : \hat{V}(X_{\gamma_k^*}) \leq g(X_{\gamma_k^*})\}$.

Then $\tau^\varepsilon \leq \tau^*$.

Martingality

Define a stopped process S_k^ε indexed by k by setting

$$S_k^\varepsilon := e^{-r(\gamma_k^* \wedge \tau^\varepsilon)} \hat{V}(X_{\gamma_k^* \wedge \tau^\varepsilon}) - c \sum_{i=1}^{k} e^{-r\gamma_i^*} 1_{\{\gamma_i^* \leq \tau^\varepsilon\}}.$$

Lemma 4.2

Suppose that $\hat{V}$ is a fixed point of $\mathscr{J}$, then the process S_k^ε is a martingale with respect to the filtration $\mathscr{F}^{\gamma^*}$.

The proof follows from the strong Markov property.

An Optimal Strategy

Theorem 4.3

Assume that $\hat{V}$ is a fixed point of $\mathscr{J}$, then $V = \hat{V}$. Moreover, the pair $(\gamma^*, \tau^\varepsilon)$ is an ε-optimal strategy.

The proofs follow from the property of the supermartingale and martingale we defined before, and the optional sampling theorem.

Theorem 4.4

If τ^* is finite a.s., the pair (γ^*, τ^*) is an optimal strategy.

$\Longrightarrow$ V is the unique fixed point of $\mathscr{J}$.

An Alternative Proof

What if we can observe the process for at most n times?
$\Longrightarrow$ We can still use the iterative method.
Define V_n to be the value function when the underlying process can be observed for at most n times:

$$V_n(x) = \sup_{\gamma \in \Gamma} \sup_{\tau \in \mathscr{S}_n^{\gamma}} \mathbb{E}[e^{-r\tau} g(X_\tau) - \sum_{i=1}^{\infty} c e^{-r\gamma_i} 1_{\{\gamma_i \le \tau\}}],$$

where $\mathscr{S}_n^{\gamma} := \{\tau \in \mathscr{S}^{\gamma} : \tau \le \gamma_n\}$, then we have

Lemma 2.3

$\lim_{n \to \infty} V_n(x) = V(x)$.

$\Longrightarrow$ For V it suffices consider at most boundedly many observations.

When number of observation is restricted

We can also prove that V is the unique fixed point of $\mathscr{J}$ by claiming

$$V_n = f_n,$$

and use the fact that

$$\lim_{n\to\infty} V_n = V,$$

and

$$\lim_{n\to\infty} f_n = f_\infty.$$

This can be done using induction and Markov property.

Outline

American Put Option as an Example

We consider the perpetual American put option for example, where the underlying process solves:

$$\begin{cases} dX_t & = \mu X dt + \sigma X dW_t \\ X_0 & = x \end{cases}$$

The value function can be written as

$$V(x) = \sup_{\hat{\tau}} \sup_{\tau \in \mathscr{S}^{\hat{\tau}}} \mathbb{E}[e^{-r\tau}(K - X_\tau)^+ - \sum_{i=1}^{\infty} c e^{-r\tau_i} 1_{\{\tau_i \le \tau\}}]$$

Upper Bound

Let us write the value of a perpetual American put option as $V^{Am} = \sup_\tau \mathbb{E}_x[e^{-r\tau}(K - X_\tau)^+]$, then $V \le V^{Am}$.

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Some properties

Define:

$$q_0 = (K - x)^+, \text{ and } q_n = \mathscr{J} q_{n-1}, \ \ n \geq 1,$$

then $q_\infty := \lim_{n\to\infty} q_n$ is the unique fixed point of $\mathscr{J}$. Furthermore:

- q_n decreases in x for all $n \geq 0$, q_∞ decreases in x.
- q_n is convex in x for all $n \geq 0$, q_∞ is convex in x.
- For all n and any $x > 0$ we have $|q_n(y) - q_n(x)| \leq \frac{K}{x}|y - x|$ for $y \in (0, \infty)$

Boundary of the Continuation Region

Lemma 5.4

$\forall n$ the continuation region has a lower bound.

Lemma 5.5

The continuation region

- has an upper bound
  when $\mu - \frac{1}{2}\sigma^2 > 0$, or when $\mu - \frac{1}{2}\sigma^2 = 0$ and $K \leq 2c$,
- has no upper bound
  when $\mu - \frac{1}{2}\sigma^2 < 0$, or when $\mu - \frac{1}{2}\sigma^2 = 0$ and $K > 2c$.

Contraction and Rate of Convergence

Lemma 5.6

The first optimal observation time is strictly bounded from 0: $\forall$ n,x,

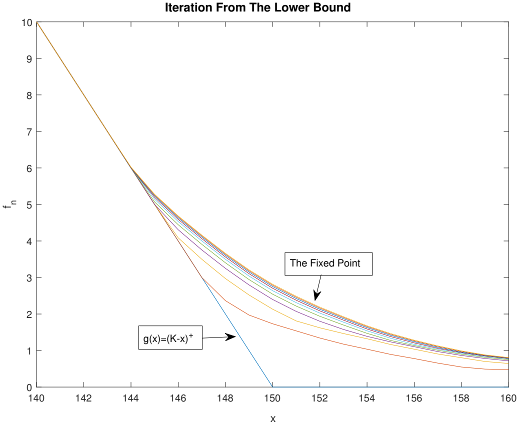
$$t_n^*(x) \geq \varepsilon$$

for some $\varepsilon > 0$.

It follows that:

- We have contraction property,
- f_n converges uniformly to f_∞,
- The rate of convergence is exponential,
- Can start from an arbitrary element to obtain the fixed point.

American Put Option as an Example



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Allowing Stopping at Any Time

We will prove: it reduces to the previous problem.
Let:

$$\mathscr{T}^\gamma = \{\tau : \mathscr{G}^\gamma \text{ - stopping times}\}$$

be the set of $\mathscr{G}^\gamma$-stopping times. Define a new value function W:

$$W : [0, \infty) \to [0, \infty)$$

$$W(x) = \sup_{\gamma \in \Gamma} \sup_{\tau \in \mathscr{T}^\gamma} \mathbb{E}\left[e^{-r\tau} g(X_\tau) - \sum_{i=1}^{\infty} c e^{-r\gamma_i} 1_{\{\gamma_i \leq \tau\}} \right].$$

Introduce a new function h:

$$h : \mathbb{R}_+ \to \mathbb{R}_+$$

$$h(x) = \sup_{t \geq 0} \mathbb{E}_x[e^{-rt} g(X_t)]$$

Allowing Stopping at Any Time

Lemma 6.1

Let $\hat{\gamma} = \{\gamma_1, \gamma_2, \dots\}$ be given, then:

$$\sup_{\tau \in \mathscr{T}^{\hat{\gamma}}} \mathbb{E}_x[e^{-r\tau} g(X_\tau) - \sum_{i=1}^{\infty} ce^{-r\gamma_i} 1_{\{\gamma_i \leq \tau\}}]$$

$$= \sup_{\tau' \in \mathscr{S}^{\hat{\gamma}}} \mathbb{E}_x[e^{-r\tau'} h(X_{\tau'})] - \sum_{i=1}^{\infty} ce^{-r\gamma_i} 1_{\{\gamma_i \leq \tau'\}}]$$

$\Longrightarrow$ Only need to consider when we are only allowed to stop immediately.

Paying the Cost In Advance

Also consider paying the cost of the next observation at the current time:

$$\tilde{V}(x) = \sup_{\gamma \in \Gamma} \sup_{\tau \in \mathscr{S}^{\gamma}} \mathbb{E}\left[e^{-r\tau} g(X_\tau) - \sum_{i=1}^{\infty} c e^{-r\gamma_{i-1}} 1_{\{\gamma_i \le \tau\}} \right].$$

Still in working progress.

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Summary

- We formulate the stopping problem where each observation comes with fixed costs;
- We define an associated operator and prove that its unique fixed point characterizes the value function. We then provide an optimal strategy where one chooses how to distribute the future observations and when to stop in terms of the fixed point.
- We then use an iterative procedure to reach the fixed point, and provide a specific example of this procedure which has an exponential convergence rate.
- We prove that allowing stopping the process at any time can be reduced to the previous problem.

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