Sequential Testing and Quickest Detection for a Multi-dimensional Wiener Process

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January 24, 2020

Another Look at the 1D case

- let Y be a continuous-time Markov chain with state space $\{0, 1\}$ and transition matrix
$$Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}$$
where $\lambda \geq 0$ is a known constant.
- $\mathbb{P}(Y_0 = 1) = \pi$ and $\mathbb{P}(Y_0 = 0) = 1 - \pi$ for $\pi \in [0, 1]$.
- let X be
$$X_t = \int_0^t Y_s \, ds + W_t,$$
where W is a 1D standard BM independent of Y.

–Can then reformulate the classical problems of sequential testing and Bayesian quickest detection.

Another Look at the 1D case

Sequential Testing

1. $\lambda = 0,\ Y_t = Y_0$,
2. Want to determine Y_0.
3. i.e. Consider

$$\inf_{\tau,d} \{\mathbb{P}(d \neq Y_0) + c\mathbb{E}[\tau]\},$$

Quickest Detection

1. $\lambda > 0$
2. Want to determine the jump time of Y_t.
3. i.e. Consider

$$\inf_{\tau} \left\{\mathbb{P}(Y_\tau = 0) + c\mathbb{E}\left[\int_0^\tau Y_t\, dt\right]\right\},$$

where τ's are $\mathscr{F}^X$-stopping times and $d \in \{0,1\}$, $\mathscr{F}_\tau^X$-measurable.

Reduction to Optimal Stopping

- Define the conditional probability process $\Pi_t := \mathbb{E}[Y_t|\mathscr{F}_t]$,
- Both problems can be written as:

$$\inf_{\tau} \mathbb{E}\left[g(\Pi_\tau) + \int_0^\tau h(\Pi_s)\, ds\right]$$

- g and h are certain penalty functions:
  - \- Testing: $g(\pi) = \pi \wedge (1-\pi)$ and $h(\pi) = c$,
  - \- Detection: $g(\pi) = (1-\pi)$ and $h(\pi) = c\pi$.

The Higher-dimensional Version

- Observe: n-dimensional BM X_t with drift
- Drift of X_i is modelded by Y^i (mutually independent) with state space $\{0,1\}$ and transition matrix

$$Q^i = \begin{pmatrix} -\lambda^i & \lambda^i \\ 0 & 0 \end{pmatrix},$$

  where $\lambda^i \geq 0$
- $\mathbb{P}(Y_0 = 1) = \pi_i \in [0,1]$
- $(X_t)_{t\geq 0} = (X_t^1, X_t^2, \ldots, X_t^n)_{t\geq 0}$ is then given by

$$dX_t^i = Y_t^i dt + dW_t^i$$

- $W^i, \ldots, W^n$ are independent BMs, Y and W are independent.
- Introduce $\Pi = (\Pi^1, \ldots \Pi^n)$:

$$\Pi_t^i := \mathbb{E}[Y_t^i | \mathscr{F}_t]$$

The Higher-dimensional Version

- We study a family of problems which can be written as:

$$\inf_{\tau} \mathbb{E}\left[g(\Pi_\tau) + \int_0^\tau h(\Pi_t)\,dt\right]$$

for $g, h : [0,1]^n \to [0,\infty)$ of the process Π.

- Assumptions
  1. One stopping rule
  2. Independence of the driving BM's
  3. g, h Lipschitz continuous
  4. g, h concave in each direction separately.

Some Formulations-Sequential Testing

Assume $\lambda_i = 0$ and $h = c$ for some $c > 0$.

- SQ1: Penalizing each faulty decision equally

$$\inf_{\tau,d} \left\{ \sum_{i=1}^{n} \mathbb{P}(d_i \neq Y_0^i) + c\mathbb{E}[\tau] \right\},$$

$$g(\pi) = \sum_{i=1}^{n} \pi_i \wedge (1 - \pi_i)$$

- SQ2: Penalized for at least one faulty decision

$$\inf_{\tau,d} \left\{ \mathbb{P}(\{d_i \neq Y_0^i \text{ for some } i\}) + c\mathbb{E}[\tau] \right\},$$

$$g(\pi) = 1 - \Pi_{i=1}^{n}(1 - \pi_i \wedge (1 - \pi_i))$$

- SQ3: Determine one drift d and point out its coordinates $\tilde{d}$

$$\inf_{\tau,d,\tilde{d}} \left\{ \mathbb{P}(d \neq Y_0^{\tilde{d}}) + c\mathbb{E}[\tau] \right\}, \quad g(\pi) = \wedge_{i=1}^{n} \pi_i \wedge (1 - \pi_i)$$

Some Formulations-Sequential Testing

- SQ4: With cost reduction, $n=2$

$$\inf_{\tau_1,\tau_2,d_1,d_2}\{\mathbb{P}(d_1\neq Y_0^1)+\mathbb{P}(d_2\neq Y_0^2)+ c\mathbb{E}[\tau_1\wedge\tau_2+(1-\lambda)(\tau_1\vee\tau_2-\tau_1\wedge\tau_2)]\}$$

- 1. Can be reduced to one stopping rule
  2. $g(\pi)=\wedge_{i=1}^{2}(\pi_i\wedge(1-\pi_i)+V_{c(1-\lambda)}\pi_{3-i})$ where $V_{c(1-\lambda)}$ is the value function of the 1D testing problem with cost $c(1-\lambda)$.
  3. when $\lambda=0$ same as SQ1, when $\lambda=1$ same as SQ3.

Some Formulations-Quickest Detection

- QD1: The *first* changing point

$$\inf_{\tau}\left\{\mathbb{P}\left(\max_{1\le i\le n} Y^i_\tau = 0\right) + c\mathbb{E}\left[\int_0^\tau \max_{1\le i\le n} Y^i_t\,dt\right]\right\}$$

$$g(\pi) = \Pi_{i=1}^n(1-\pi_i),\quad h(\pi) = c(1-\Pi_{i=1}^n(1-\pi_i))$$

- QD2: The *last* changing point

$$\inf_{\tau}\left\{\mathbb{P}\left(\min_{1\le i\le n} Y^i_\tau = 0\right) + c\mathbb{E}\left[\int_0^\tau \min_{1\le i\le n} Y^i_t\,dt\right]\right\}$$

$$g(\pi) = 1-\Pi_{i=1}^n \pi_i,\quad h(\pi) = c\Pi_{i=1}^n \pi_i$$

- QD3: Determine one change point and point out its coordinates $\tilde{d}$

$$\inf_{\tau,\tilde{d}}\left\{\mathbb{P}(Y^{\tilde{d}}_\tau = 0) + c\mathbb{E}\left[\int_0^\tau \sum_{i=1}^n Y^i_t\,dt\right]\right\},$$

$$g(\pi) = \wedge_{i=1}^n(1-\pi_i),\quad h(\pi) = c\sum_{i=1}^n \pi_i$$

Main Properties

Unilateral Concavity

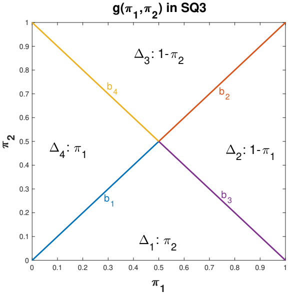
The function $\pi_i \mapsto V(\pi)$ is concave in each variable separately (i.e. $\pi_i \mapsto V(\pi)$ is concave for each $i = 1, ..., n$).

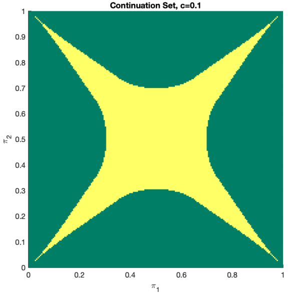
Some Other Results

1. The cost function V is Lipschitz continuous.
2. The set $\{\mathscr{L}g + h < 0\}$ is contained in $\mathscr{C}$.
3. The kinks in the g function are contained in $\mathscr{C}$.

SQ3: Testing with Pointing the Direction

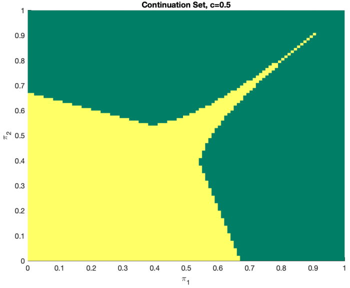
1. $g(\pi) = \wedge_{i=1}^{n} \pi_i \wedge (1 - \pi_i)$
2. The diagonals are contained in $\mathscr{C}$
3. The square $[A_*, 1 - A_*] \times [A_*, 1 - A_*]$ is contained in the continuation region.





QD3: Detection with Pointing out the Direction

1. $g(\pi) = 1 - \pi_1 \vee \pi_2$
2. The diagonal $\{\pi_1 = \pi_2\}$ is contained in $\mathscr{C}$,
3. One can find a crude upper bound for $\mathscr{C}$: $\pi_2 < \frac{\lambda}{c} - (\frac{\lambda}{c} + 1)\pi_1$,
4. By connecting with the 1D problem, one can show that the stopping region intersects π_1 axis at some $\exists C^* \in [\frac{\lambda}{\lambda+c}, 1)$.



Summary

- **Reformulate** the 1D testing and detection problems as one,
- **Extend it** into a family of *multi-dimensional* stopping problems with *one stopping rule* and *independence* in the driving Brownian motions,
- **Prove unilateral concavity** and some other general properties,
- **Give some formulations** in this family possibly with applications.