

# Stopping problems with an unknown state

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# Motivation: an example

- A stopper does not always have full information
- Consider the underlying with two possible states: "good/bad", e.g.,

$$dX_t = \theta dt + dW_t,$$

where  $\theta \in \{-1, 1\}$

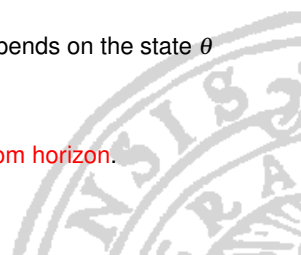
- We stop with competition:

$$\sup_{\tau} \mathbb{E}[\theta 1_{\tau < \gamma}]$$

where  $\gamma$  is when your competition stops.

- Opportunities to stop would disappear, the rate depends on the state  $\theta$
- Competitor has not stopped yet! –information on  $\theta$

We study stopping problems with **state-dependent random horizon**.



# Problem formulation

- Consider Bernoulli random variable  $\theta$

$$\mathbb{P}(\theta = 1) = \pi = 1 - \mathbb{P}(\theta = 0),$$

and Brownian motion  $W$  independent of  $\theta$ .

- Let random time  $\gamma$  depend on  $\theta$ , and be independent of  $W$ :

$$\mathbb{P}(\gamma > t | \theta = i) = F_i(t),$$

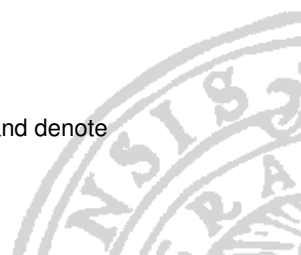
where  $F_i$  continuous, non-increasing,  $F_i(0) = 1$ .

- Let the underlying  $X$  depend on  $\theta$ :

$$dX_t = \mu(X_t, \theta)dt + \sigma(X_t)dW_t,$$

and denote  $\mu_i(x) = \mu(x, i)$ .

- Let the payoff  $g : [0, \infty) \times \mathbb{R} \times \{0, 1\}$  depend on  $\theta$ , and denote  $g_i(t, x) = g(t, x, i)$ .



# Problem formulation

- We consider the following problem:

$$V = \sup_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[ g(\tau, X_\tau, \theta) 1_{\{\tau < \gamma\}} \right]. \quad (1)$$

- $\mathcal{T}^{X,\gamma}$ : the set of  $\mathcal{F}^{X,\gamma}$ -stopping times,
- $\mathcal{F}^{X,\gamma}$ : generated by  $X$  and  $1_{\cdot \geq \gamma}$ .

- Note that

- $g(t, x, \theta) = g(t, \theta)$ : statistical problems,  
 *$X$  serves as an observation process*
- $g(t, x, \theta) = g(t, x)$ : financial problems,  
 *$\theta$  implicitly affect the payoff through  $X$*



# Incomplete to complete information

- Observe that:

$$v = \sup_{\tau \in \mathcal{T}^X} \mathbb{E}_\pi \left[ g(\tau, X_\tau, \theta) 1_{\{\tau < \gamma\}} \right] = V. \quad (2)$$

- Define the conditional probability process:

$$\Pi_t := \mathbb{P}_\pi(\theta = 1 | \mathcal{F}_t^X)$$

## Proposition

*We have*

$$V = \sup_{\tau \in \mathcal{T}^X} \mathbb{E}_\pi [g_0(\tau, X_\tau)(1 - \Pi_\tau)F_0(\tau) + g_1(\tau, X_\tau)\Pi_\tau F_1(\tau)]. \quad (3)$$

*Moreover, if  $\tau \in \mathcal{T}^X$  is optimal in (2), then it is also optimal in (1).*

# Incomplete to complete information

- The pair  $(X, \Pi)$  satisfies:

$$\begin{cases} dX_t = (\mu_0(X_t) + (\mu_1(X_t) - \mu_0(X_t))\Pi_t) dt + \sigma(X_t)d\hat{W}_t \\ d\Pi_t = \omega(X_t)\Pi_t(1 - \Pi_t)d\hat{W}_t, \end{cases}$$

where  $\omega(x) = (\mu_1(x) - \mu_0(x))/\sigma(x)$ .

- The process

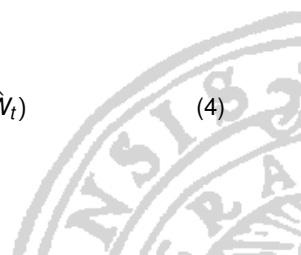
$$\hat{W}_t := \int_0^t \frac{dX_s}{\sigma(X_s)} - \int_0^t \frac{1}{\sigma(X_s)} (\mu_0(X_s) + (\mu_1(X_s) - \mu_0(X_s))\Pi_s) ds$$

is a  $\mathbb{P}_\pi$ -Brownian motion.

- The process  $\Phi := \frac{\Pi_t}{1 - \Pi_t}$  satisfies

$$d\Phi_t = \omega(X_t)\Phi_t(\omega(X_t)\Pi_t dt + d\hat{W}_t) \quad (4)$$

with initial condition  $\Phi_0 = \varphi := \pi/(1 - \pi)$ .



# A measure change

## Lemma

Given a stopping time  $\tau \in \mathcal{T}^X$ , denote by  $\mathbb{P}_{\pi, \tau}$  the measure  $\mathbb{P}_{\pi}$  restricted to  $\mathcal{F}_{\tau}$ ,  $\pi \in [0, 1]$ . We then have

$$\frac{d\mathbb{P}_{0, \tau}}{d\mathbb{P}_{\pi, \tau}} = \frac{1 + \Phi_{\tau}}{1 + \varphi}.$$

- Under  $\mathbb{P}^0$ ,  $(X, \Phi)$  satisfies

$$\begin{cases} dX_t = \mu_0(X_t) dt + \sigma(X_t) dW_t \\ d\Phi_t = \omega(X_t) \Phi_t dW_t \end{cases} \quad (5)$$

- Introduce the process

$$\Phi_t^{\circ} := \frac{F_1(t)}{F_0(t)} \Phi_t, \quad (6)$$

- $\Phi_t^{\circ}$  satisfies

$$d\Phi_t^{\circ} = \frac{f'(t)}{f(t)} \Phi_t^{\circ} dt + \omega(X_t) \Phi_t^{\circ} dW_t.$$

# A measure change

## Theorem

Denote by

$$U = \sup_{\tau \in \mathcal{T}^X} \mathbb{E}_\varphi^0 [F_0(\tau) (g_0(\tau, X_\tau) + g_1(\tau, X_\tau) \Phi_\tau^\circ)], \quad (7)$$

Then  $V = U/(1 + \varphi)$ , where  $\varphi = \pi/(1 - \pi)$ . Moreover, if  $\tau \in \mathcal{T}^X$  is an optimal stopping in (7), then it is also optimal in the original problem (1).





# 1. A hiring problem

- Hire a person, good/bad:

$$X_t = \mu(\theta)t + \sigma W_t$$

- Benefit of hiring:

$$g(t, x, \theta) = \begin{cases} -e^{-rt}c & \text{if } \theta = 0 \\ e^{-rt}d & \text{if } \theta = 1 \end{cases}$$

- Survival probabilities: exponential

$$F_0(t) = e^{-\lambda_0 t} \quad \& \quad F_1(t) = e^{-\lambda_1 t},$$

- The stopping problem:

$$V = \sup_{\tau \in \mathcal{T}^{X,Y}} \mathbb{E}_\pi \left[ e^{-r\tau} \left( d \mathbf{1}_{\{\theta=1\}} - c \mathbf{1}_{\{\theta=0\}} \right) \mathbf{1}_{\{\tau < \gamma\}} \right],$$

where  $\pi = \mathbb{P}_\pi(\theta = 1)$ .

# 1. A hiring problem

- Rewrite:

$$V = \frac{1}{1 + \varphi} \sup_{\tau \in \mathcal{T}^X} \mathbb{E}^0 \left[ e^{-(r+\lambda_0)\tau} (\Phi_\tau^\circ d - c) \right],$$

where  $\Phi_t^\circ$  is a GBM:

$$d\Phi_t^\circ = -(\lambda_1 - \lambda_0)\Phi_t^\circ dt + \omega\Phi_t^\circ dW.$$

- The value function:

$$V = \frac{d}{1 + \varphi} \sup_{\tau \in \mathcal{T}^X} \mathbb{E}^0 \left[ e^{-(r+\lambda_0)\tau} \left( \Phi_\tau^\circ - \frac{c}{d} \right) \right] = \frac{d}{1 + \varphi} V^{Am}(\varphi).$$

- $V^{Am}$  is the value of the American call option with underlying  $\Phi^\circ$  and strike  $\frac{c}{d}$ : **explicit**.

## 2. Sequential testing with random horizon

- Let  $X_t = \theta t + \sigma W_t$ .
- Consider a sequential testing problem of minimising

$$\mathbb{P}(\theta \neq d) + c\mathbb{E}[\tau]$$

with random horizon.

- where  $F_1(t) = 1$  and  $F_0(t) = e^{-\lambda t}$ .
- The value function

$$V = \inf_{\tau \in \mathcal{F}^{X,\gamma}} \mathbb{E}[\Pi_\tau^\circ \wedge (1 - \Pi_\tau^\circ) + c\tau],$$

where

$$\Pi_t^\circ := \mathbb{P}(\theta = 1 | \mathcal{F}_t^{X,\gamma}).$$

## 2. Sequential testing with random horizon

- Rewrite

$$V = \frac{1}{1+\varphi} \inf_{\tau \in \mathcal{T}^X} \mathbb{E}^0 \left[ F_0(\tau) (\Phi_\tau^\circ \wedge 1) + c \int_0^\tau F_0(t) (1 + \Phi_t^\circ) dt \right]$$

- where

$$d\Phi_t^\circ = \lambda \Phi_t^\circ dt + \omega \Phi_t^\circ dW_t^0.$$

- Define the blue part as  $U(\varphi)$ ,

$$\begin{cases} \frac{1}{2} \omega^2 \varphi^2 U_{\varphi\varphi} + \lambda \varphi U_\varphi - \lambda U + c(1 + \varphi) = 0, & \varphi \in (A, B) \\ U(A) = A, U_\varphi(A) = 0 \\ U(B) = 1, U_\varphi(B) = 1 \end{cases}$$

## 3. Real options with competition

- Consider a GBM:

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

- and the stopping problem

$$\sup_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}[e^{-r\tau}(X_\tau - K)^+ 1_{\{\tau < \gamma\}}].$$

- $P(\gamma > t) = e^{-\lambda t}$  where
  - $\lambda = 0$  on  $\{\theta = 0\}$ ,
  - $\lambda = \lambda_1$  on  $\{\theta = 1\}$ .



## 3. Real options with competition

- Rewrite

$$V = \frac{1}{1 + \varphi} \sup_{\tau \in \mathcal{T}^X} \mathbb{E}_{X, \varphi}^0 [e^{-r\tau} (X_\tau - K)^+ (1 + \Phi_\tau^\circ)]$$

- where  $\Phi_t^\circ = e^{-\lambda_1 t} \varphi$ .
- We can characterise the boundary  $b(\varphi)$ :

$$0 = (b(\varphi) - K)(1 + \varphi) + \mathbb{E}_{b(\varphi), \varphi}^0 \left[ \int_0^\infty e^{-rt} \mathcal{L}V(X_t, \Phi_t^\circ) \mathbf{1}_{\{X_t > b(\varphi)\}} dt \right],$$

- Get an integral equation with normal CDFs.



# Thank you!

