Stopping problems with an unknown state

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Motivation: an example

- A stopper does not always have full information
- Consider the underlying with two possible states: "good/bad", e.g.,

$$dX_t = \theta dt + dW_t,$$

  where $\theta \in \{-1, 1\}$
- We stop with competition:

$$\sup_{\tau} \mathbb{E}[\theta 1_{\tau<\gamma}]$$

  where γ is when your competition stops.
- Opportunities to stop would disappear, the rate depends on the state θ
- Competitor has not stopped yet! –information on θ

We study stopping problems with state-dependent random horizon.

Problem formulation

- Consider Bernoulli random variable θ

$$\mathbb{P}(\theta = 1) = \pi = 1 - \mathbb{P}(\theta = 0),$$

and Brownian motion W independent of θ.

- Let random time γ depend on θ, and be independent of W:

$$\mathbb{P}(\gamma > t | \theta = i) = F_i(t),$$

where F_i continuous, non-increasing, $F_i(0) = 1$.

- Let the underlying X depend on θ:

$$dX_t = \mu(X_t, \theta)dt + \sigma(X_t)dW_t,$$

and denote $\mu_i(x) = \mu(x, i)$.

- Let the payoff $g : [0, \infty) \times \mathbb{R} \times \{0, 1\}$ depend on θ, and denote $g_i(t, x) = g(t, x, i)$.

Problem formulation

- We consider the following problem:

$$V = \sup_{\tau \in \mathscr{T}^{X,\gamma}} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau < \gamma\}} \right]. \quad (1)$$

– $\mathscr{T}^{X,\gamma}$: the set of $\mathscr{F}^{X,\gamma}$-stopping times,
– $\mathscr{F}^{X,\gamma}$: generated by X and $1_{\cdot \geq \gamma}$.

- Note that
  - $g(t,x,\theta) = g(t,\theta)$: statistical problems, X serves as an observation process
  - $g(t,x,\theta) = g(t,x)$: financial problems, θ implicitly affect the payoff through X

Incomplete to complete information

- Observe that:

$$v = \sup_{\tau \in \mathscr{T}^X} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau < \gamma\}} \right] = V. \quad (2)$$

- Define the conditional probability process:

$$\Pi_t := \mathbb{P}_\pi(\theta = 1 | \mathscr{F}_t^X)$$

Proposition

We have

$$V = \sup_{\tau \in \mathscr{T}^X} \mathbb{E}_\pi \left[g_0(\tau, X_\tau)(1 - \Pi_\tau) F_0(\tau) + g_1(\tau, X_\tau) \Pi_\tau F_1(\tau) \right]. \quad (3)$$

Moreover, if $\tau \in \mathscr{T}^X$ is optimal in (2), then it is also optimal in (1).

Incomplete to complete information

- The pair (X, Π) satisfies:

$$\begin{cases} dX_t = (\mu_0(X_t) + (\mu_1(X_t) - \mu_0(X_t))\Pi_t)\,dt + \sigma(X_t)d\hat{W}_t \\ d\Pi_t = \omega(X_t)\Pi_t(1-\Pi_t)\,d\hat{W}_t, \end{cases}$$

where $\omega(x) = (\mu_1(x) - \mu_0(x))/\sigma(x)$.

- The process

$$\hat{W}_t := \int_0^t \frac{dX_t}{\sigma(X_s)} - \int_0^t \frac{1}{\sigma(X_t)}(\mu_0(X_s) + (\mu_1(X_s) - \mu_0(X_s))\Pi_s)\,ds$$

is a $\mathbb{P}_\pi$-Brownian motion.

- The process $\Phi := \frac{\Pi_t}{1-\Pi_t}$ satisfies

$$d\Phi_t = \omega(X_t)\Phi_t(\omega(X_t)\Pi_t\,dt + d\hat{W}_t) \tag{4}$$

with initial condition $\Phi_0 = \varphi := \pi/(1-\pi)$.

A measure change

Lemma

Given a stopping time $\tau \in \mathscr{T}^X$, denote by $\mathbb{P}_{\pi,\tau}$ the measure $\mathbb{P}_\pi$ restricted to $\mathscr{F}_\tau$, $\pi \in [0,1]$. We then have

$$\frac{d\mathbb{P}_{0,\tau}}{d\mathbb{P}_{\pi,\tau}} = \frac{1+\Phi_\tau}{1+\varphi}.$$

- Under $\mathbb{P}^0$, (X,Φ) satisfies

$$\begin{cases} dX_t = \mu_0(X_t)\,dt + \sigma(X_t)\,dW_t \\ d\Phi_t = \omega(X_t)\Phi_t\,dW_t \end{cases} \tag{5}$$

- Introduce the process

$$\Phi_t^\circ := \frac{F_1(t)}{F_0(t)}\Phi_t, \tag{6}$$

- Φ_t° satisfies

$$d\Phi_t^\circ = \frac{f'(t)}{f(t)}\Phi_t^\circ\,dt + \omega(X_t)\Phi_t^\circ dW_t.$$

A measure change

Theorem

Denote by

$$U = \sup_{\tau \in \mathscr{T}^X} \mathbb{E}^0_\varphi [F_0(\tau)(g_0(\tau, X_\tau) + g_1(\tau, X_\tau)\Phi^\circ_\tau)], \qquad (7)$$

Then $V = U/(1+\varphi)$, *where* $\varphi = \pi/(1-\pi)$. *Moreover, if* $\tau \in \mathscr{T}^X$ *is an optimal stopping in* (7)*, then it is also optimal in the original problem* (1)*.*

1. A hiring problem

- Hire a person, good/bad:

$$X_t = \mu(\theta)t + \sigma W_t$$

- Benefit of hiring:

$$g(t,x,\theta) = \begin{cases} -e^{-rt}c & \text{if } \theta = 0 \\ e^{-rt}d & \text{if } \theta = 1 \end{cases}$$

- Survival probabilities: exponential

$$F_0(t) = e^{-\lambda_0 t} \qquad \& \qquad F_1(t) = e^{-\lambda_1 t},$$

- The stopping problem:

$$V = \sup_{\tau \in \mathscr{T}^{X,\gamma}} \mathbb{E}_\pi \left[e^{-r\tau} \left(d1_{\{\theta=1\}} - c1_{\{\theta=0\}} \right) 1_{\{\tau<\gamma\}} \right],$$

where $\pi = \mathbb{P}_\pi(\theta = 1)$.

1. A hiring problem

- Rewrite:

$$V = \frac{1}{1+\varphi} \sup_{\tau \in \mathcal{T}^X} \mathbb{E}^0 \left[e^{-(r+\lambda_0)\tau} (\Phi^\circ_\tau d - c) \right],$$

where Φ°_t is a GBM:

$$d\Phi^\circ_t = -(\lambda_1 - \lambda_0)\Phi^\circ_t\, dt + \omega \Phi^\circ_t\, dW.$$

- The value function:

$$V = \frac{d}{1+\varphi} \sup_{\tau \in \mathcal{T}^X} \mathbb{E}^0 \left[e^{-(r+\lambda_0)\tau} \left(\Phi^\circ_\tau - \frac{c}{d} \right) \right] = \frac{d}{1+\varphi} V^{Am}(\varphi).$$

- V^{Am} is the value of the American call option with underlying Φ° and strike $\frac{c}{d}$: explicit.

2. Sequential testing with random horizon

- Let $X_t = \theta t + \sigma W_t$.
- Consider a sequential testing problem of minimising

$$\mathbb{P}(\theta \neq d) + c\mathbb{E}[\tau]$$

with random horizon.

- where $F_1(t) = 1$ and $F_0(t) = e^{-\lambda t}$.
- The value function

$$V = \inf_{\tau \in \mathscr{T}^{X,\gamma}} \mathbb{E}\left[\Pi^\circ_\tau \wedge (1 - \Pi^\circ_\tau) + c\tau\right],$$

where

$$\Pi^\circ_t := \mathbb{P}(\theta = 1 | \mathscr{F}^{X,\gamma}_t).$$

2. Sequential testing with random horizon

- Rewrite

$$V = \frac{1}{1+\varphi} \inf_{\tau \in \mathscr{T}^X} \mathbb{E}^0 \left[F_0(\tau)(\Phi^\circ_\tau \wedge 1) + c \int_0^\tau F_0(t)(1+\Phi^\circ_t)\, dt \right]$$

- where

$$d\Phi^\circ_t = \lambda \Phi^\circ_t\, dt + \omega \Phi^\circ_t dW^0_t.$$

- Define the blue part as $U(\varphi)$,

$$\begin{cases} \frac{1}{2}\omega^2\varphi^2 U_{\varphi\varphi} + \lambda\varphi U_\varphi - \lambda U + c(1+\varphi) = 0, & \varphi \in (A,B) \\ U(A) = A, U_\varphi(A) = 0 & \\ U(B) = 1, U_\varphi(B) = 1 & \end{cases}$$

3. Real options with competition

- Consider a GBM:

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

- and the stopping problem

$$\sup_{\tau \in \mathscr{T}^{X,\gamma}} \mathbb{E}[e^{-r\tau}(X_\tau - K)^+ 1_{\{\tau<\gamma\}}].$$

- $P(\gamma > t) = e^{-\lambda t}$ where
  - $\lambda = 0$ on $\{\theta = 0\}$,
  - $\lambda = \lambda_1$ on $\{\theta = 1\}$.

3. Real options with competition

- Rewrite

$$V = \frac{1}{1+\varphi} \sup_{\tau \in \mathcal{T}^X} \mathbb{E}^0_{x,\varphi}\left[e^{-r\tau}(X_\tau - K)^+ (1 + \Phi^\circ_\tau)\right]$$

- where $\Phi^\circ_t = e^{-\lambda_1 t}\varphi$.
- We can characterise the boundary $b(\varphi)$:

$$0 = (b(\varphi) - K)(1+\varphi) + \mathbb{E}^0_{b(\varphi),\varphi}\left[\int_0^\infty e^{-rt}\mathcal{L}V(X_t, \Phi^\circ_t) 1_{\{X_t > b(\Phi^\circ_t)\}} dt\right],$$

- Get an integral equation with normal CDFs.

Thank you!