Asymmetric Dynkin ghost games with consolation

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Outline

Ghost games with preemption

- Consider a two-player non-zero sum Dynkin game (**De Anglis and Ekström, 2020)**
- **Key feature:** each player is uncertain about the existence of a competitor.

$$\theta_i = \text{"Player i has competition"}$$

- **Preemption:** the first one to stop at time t gets $g(X_t)$, the second gets nothing
- Assume that Player 1 stops at τ, Player 2 stops at γ, their rewards:

$$R_1(\tau,\gamma) := (g(X_\tau)1_{\tau\leq\hat{\gamma}})1_{\tau<\infty},$$

$$R_2(\tau,\gamma) := (g(X_\gamma)1_{\gamma<\hat{\tau}})1_{\gamma<\infty},$$

- where

$$\hat{\gamma} = \gamma 1_{\theta_1=1} + \infty 1_{\theta_1=0}$$

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- **Introduction**
- Our ghost game
- Examples

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Ghost games with preemption

- At the start of the game, each player estimates their probability of competition:

$$\mathbb{P}(\theta_i = 1) = p_i.$$

Then they adjust their belief processes $\Pi^i_t = \mathbb{P}(\theta_i = 1 | \mathscr{F}^X_t, \hat{\gamma} > t)$ by observing:

  - the underlying X,
  - the lack of action of their competitor.

- Note that we can "fool" our competitor,
- A pure-strategy equilibrium wouldn't exist!

This means τ, γ should be randomised stopping times:

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Furthermore, Π_t^i is a function of Γ_t^{3-i}:

$$\Pi_t^i = \frac{p_i(1-\Gamma_t^{3-i})}{1-p_i\Gamma_t^{3-i}}.$$

Wlog, assume $p_1 < p_2$.

- The players seek to maximise their discounted expected payoff:

$$J_1(\tau,\gamma,p_1,x) := \mathbb{E}_x[e^{-r\tau}R_1(\tau,\gamma)],$$

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- The pair (τ^*,γ^*) is a Nash Equilibrium if

$$J_1(\tau,\gamma^*,p_1,x) \le J_1(\tau^*,\gamma^*,p_1,x),$$

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Some results they found:

- A Nash Equilibrium exits
- $d\Gamma_t^{1,*} = \frac{p_1}{p_2} d\Gamma_t^{2,*}$
- $u_1(p_1, x) = u_2(p_2, x) = (1 - p_1)V^g(x)$,

where V^g is the "American value" of a single player.

How does the process (X, Π^i) behave?

- Getting pushed along the stopping boundary
- Jump to 0 after the competitor is revealed.

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Our generalisation

We generalise this game in two aspects:

- asymmetry in the immediate payoff g,
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- Let the underlying X be a continuous strong Markov process,
- Assume $g_i, h_i : \mathbb{R} \to [0, \infty)$, $i = 1, 2$, are given continuous functions with $g_i \geq h_i$
- Denote $V^{g_i}(x)$, $V^{h_i}(x)$ as the "American values",
- The expected discounted payoff:

$$J_1(x; \gamma_1, \gamma_2) := \mathbb{E}_x[e^{-r\gamma_1} g_1(X_{\gamma_1}) 1_{\{\gamma_1 \leq \hat{\gamma}_2\}} + e^{-r\gamma_2} V^{h_1}(X_{\gamma_2}) 1_{\{\gamma_1 > \hat{\gamma}_2\}}],$$

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Some results

Proposition

If γ_2 is a $(U, \Gamma^2)-$randomised stopping time and τ is a stopping time, then

$$J_1(x;\tau,\gamma_2) = \mathbb{E}_x\left[e^{-r\tau}g_1(X_\tau)(1-p_1\Gamma^2_\tau)+p_1\int_{[0,\tau)}e^{-rt}V^{h_1}(X_t)\,d\Gamma^2_t\right].$$

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Theorem

Let two continuous functions $u^1, u^2 : \mathbb{R} \times [0,1]^2 \to [0,\infty)$ *and a pair* (Γ^1, Γ^2) *be given. Define two processes*

$$M_t^1 := e^{-rt}(1-p_1\Gamma_t^2)u^1(X_t,\Pi_t^1,\Pi_t^2) + p_1\int_{[0,t]} e^{-rs}V^{h_1}(X_s)\,d\Gamma_s^2,$$

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Assume that for $i = 1,2$*,*

(i) M^i *is a supermartingale,* M^1 *is a martingale for* $t \le \tau_{g_1}$*, and* M^2 *is a martingale for* $t < \tau_{g_2}$*;*

(ii) $u^1(X_t,\Pi_t^1,\Pi_t^2) \ge g_1(X_t)$ *and* $u^2(X_t,\Pi_t^1,\Pi_t^2) \ge g_2(X_t)$ $\mathbb{P}_x$*-a.s.;*

(iii) $\Gamma_t^i = \int_0^t 1_{\{u^i(X_s,\Pi_s^1,\Pi_s^2)=g^i(X_s)\}}\,d\Gamma_s^i$.

Then (Γ^1,Γ^2) *is a Nash equilibrium, and the equilibrium values are given by* $u^1(x,p_1,p_2)$ *and* $u^2(x,p_1,p_2)$*, respectively.*

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(ii) *$u^1(X_t, \Pi_t^1, \Pi_t^2) \ge g_1(X_t)$ and $u^2(X_t, \Pi_t^1, \Pi_t^2) \ge g_2(X_t)$ $\mathbb{P}_x$-a.s.;*

(iii) *$\Gamma_t^i = \int_0^t 1_{\{u^i(X_s, \Pi_s^1, \Pi_s^2) = g^i(X_s)\}}\, d\Gamma_s^i$.*

Then (Γ^1, Γ^2) is a Nash equilibrium, and the equilibrium values are given by $u^1(x, p_1, p_2)$ and $u^2(x, p_1, p_2)$, respectively.

Outline

Example 1. Symmetric case with no consolation

In this case $g_1 = g_2 = g$, $h_1 = h_2 = 0$, **(De Anglis and Ekström)**

- $u^i(x, p_1, p_2) = (1 - p_1)V^g(x)$, $b(x) := 1 - \frac{g(x)}{V^g(x)}$.
- Γ^2 is characterised by the boundary b and $\inf b(X_t)$
- $$\Gamma^1_t = \begin{cases} \frac{p_1}{p_2}\Gamma^2_t & t < \tau_g \\ 1 & t \geq \tau_g. \end{cases}$$
- In this case, $M^i_t = (1 - p_i)e^{-rt}V^g(X_t)$, martingales.

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Example 2. Symmetric case with consolation (special case)

In this case $g_1 = g_2 = g$, $h_1 = h_2 = h$, and

$$\{x \in \mathbb{R} : V^g(x) < g(x)\} \subseteq \{x \in \mathbb{R} : V^h(x) < h(x)\},$$

- $e^{-rt\wedge\tau^g} V^h(X_{t\wedge\tau^g})$ is a martingale,
- $u^i(x, p_1, p_2) := (1 - p_1)V^g(x) + p_1 V^h(x)$.
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- We assume that $g_i(x) = (x - K_i)^+, \quad h_i(x) = 0$, where $0 < K_2 < K_1 < b_2 < b_1$.
- The players observe a GBM

Observe that

- Player 1 is not afraid of competition
- Player 1 naturally has a larger τ^{g_1}.

These suggest an ansatz for player 1:

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- The stopping boundary for player 1:

$$p_1 = b(x) = \begin{cases} 1 - (\frac{g_1}{V^{g_1}})(x), x < b_2, \\ 0, x \geq b_2. \end{cases}$$

- $b(x)$ is non-increasing.
- Define $\hat{p} = 1 - (\frac{g_1}{V^{g_1}})(b_2)$

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- **Case 1:** $p_1 \leq \hat{p}$: both players just wait.
- **Case 2:** $p_1 > \hat{p}$:
  - On b, $u_2 = g_2$:

$$u^2(x, p_1, p_2) = \psi(x)\left(\frac{g_2}{\psi}\right)(b^{-1}(p_1)),$$

  - We have $M_t^1 = (1 - p_1)e^{-rt}V^{g_1}(X_t)$,
  - For M^2 to be a martingale, we need

$$dM_t^2 = 0$$

$$\iff d\left(\log\left(1 - p_2\Gamma_t^1\right)\right) = C(p_1)d\left(\frac{1}{1 - p_1\Gamma_t^1}\right).$$

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In this case $g_1 = g_2 = g, h_1 = h_2 = h$. $e^{-rt}V^h(X_t)$ is not necessarily a m.g.

- For now assume $p_1 = p_2 = p$.
- Consider an Ito diffusion as the underlying.
- In the x coordinate, $\mathscr{L}^x u - ru = 0, \implies u(x,p) = c(p)\psi(x)$.

On the stopping boundary: $u(x,p) = g(x)$, and the process

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When the boundary is one-sided: solve this ODE:

Theorem

Assume that $\{V^g > g\} \cap \{V^h < g\} = (x_1, x_0)$, where x_1, x_0 are the unique roots of $(V^h - g)(x) = 0$ and $(V^g - g)(x) = 0$, respectively. Assume further that $\frac{g}{\psi}$ is strictly increasing in x on the interval (x_1, x_0). Then the stopping boundary b is monotonically decreasing on (x_1, x_0). Furthermore, b has the following explicit expression:

$$b(x) = 1 - \exp\left(\int_x^{x_0} \frac{(\frac{g}{\psi})_x \psi}{V^h - g}(y)dy\right)$$

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Summary

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- In some cases (when?), we can hope for explicit solutions for one of the players
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Summary

- In the presence of asymmetry and consolation, in general, we don't have explicit solutions.
- In some cases (when?), we can hope for explicit solutions for one of the players
- The stopping boundary is a surface $f(x, p_1, p_2) = 0$.
- How to construct?
  - Solvability of variational inequality
  - Fixed-point approach?

Thank you!