Bayesian sequential testing and estimation in discrete time

Yuqiong Wang

Partly joint work with Erik Ekström.

Department of Mathematics
Uppsala University

Financial/Actuarial Mathematics Seminar @ Umich

Outline

Statistics & Optimal stopping

Classical problem: Testing the unknown drift of a BM.

- Observe the trajectory of a BM with unknown drift:

$$X_t = \Theta t + W_t.$$

where $\mathbb{P}(\Theta = 1) = \pi = 1 - \mathbb{P}(\Theta = 0)$, $\pi \in (0,1)$.

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- The time to stop observing is part of the decision.

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$$V(\pi) = \inf_\tau \mathbb{E}_\pi[c\tau + \Pi_\tau \wedge (1-\Pi_\tau)]$$

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- Standard method: explicit solution: Shiryaev (1969).

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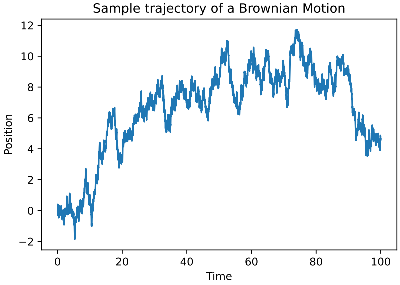

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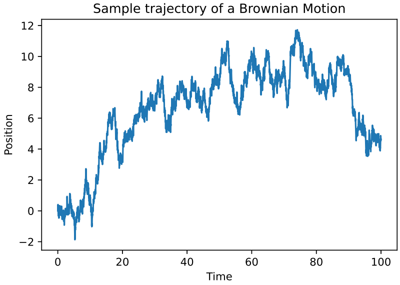

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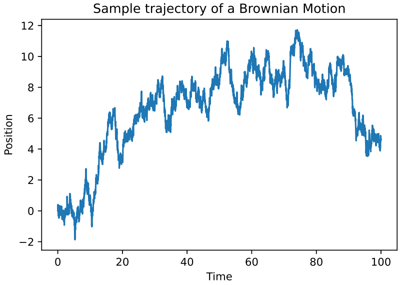

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This motivates us to consider things in **discrete time**.

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- Explicit solutions?
- Otherwise, what structural properties does the problem exhibit?

What are we building on?

Discrete time: popular in the 60s and 70s

- Studied on a case-by-case basis and rely on conjugate priors: Lindley and Barnett (1965), Moriguti and Robbins (1962).
- Focus on asymptotic behaviour: Schwartz (1962), Bickel (1973), Lai (1988).
- c.f. Sobel (1953), Alvo (1977), Cablio (1977).

Continuous time: many generalisations

- Finite horizon: Gapeev and Peskir (2004), Poisson: Peskir and Shiryaev (2000), multi-dimensional: Ekström and Wang (2022).
- Most literature uses binary priors (c.f. Zhitlukhin and Shiryaev (2011), Ekström and Vaicenavicius (2015), Ekström, Karatzas & Vaicenavicius (2022)).

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- μ: (arbitrary) prior of the unknown parameter Θ
- Denote the support of μ by S, and $S^+ := S \cap \theta_0, \infty)$.

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Can we do Markovian embedding?

Properties of the Π process

- At time n, given $X_1 = x_1, \ldots, X_n = x_n$, by the Bayes theorem,

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- Denoting $Y_n := \sum_{i=1}^n X_i$:

$$\Pi_n = q(n, Y_n).$$

where

$$q(n, y) := \frac{\int_{S^+} e^{uy - nB(u)}\mu(du)}{\int_S e^{uy - nB(u)}\mu(du)}.$$

Parameterize the posterior distribution

Denote by

$$\mu_{n,y}(du) := \frac{e^{uy-nB(u)}\mu(du)}{\int_S e^{uy-nB(u)}\mu(du)}$$

the posterior distribution of Θ at time n conditional on $Y_n = y$.

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Lemma (Dynamic programming)

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Lemma (Preservation of concavity)

Let $f : [0,1] \to [0,\infty)$ be a concave function. Then $\pi \mapsto \mathbb{E}_{n,\pi}[f(\Pi_{n+1})]$ is concave on $(0,1)$.

Main result 1: concavity

Theorem (Concavity)

The function $\pi \mapsto V(n,\pi)$ is concave for each fixed $n \geq 0$.

Introduce now

- The continuation region $\mathscr{C}$:

$$\mathscr{C} := \{(n,\pi) \in \mathbb{N}_0 \times [0,1] : V(n,\pi) < \pi \wedge (1-\pi)\},$$

- The stopping region $\mathscr{D}$ by

$$\mathscr{D} := \{(n,\pi) \in \mathbb{N}_0 \times [0,1] : V(n,\pi) = \pi \wedge (1-\pi)\}.$$

- The stopping time

$$\tau^* := \inf\{k \geq 0 : (n+k, \Pi_{n+k}) \in \mathscr{D}\}$$

is an optimal strategy.

Main result 1: concavity

The continuation region is of the form $(b_1(n), b_2(n))$

Main result 2: concentration of the posterior

The posterior distribution squeezes in

If $a < \theta_0 < b$, then

$$n \mapsto \mathbb{P}_{n,\pi}(\Theta \leq a) \quad \& \quad n \mapsto \mathbb{P}_{n,\pi}(\Theta > b)$$

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As a consequence, the $\pi-$level curves are spreading out.

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An assumption

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Conjecture: V is non-decreasing in n.

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Introduction

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What is the value of the unknown parameter?

- Want to obtain an accurate estimate in the presence of cost.
- We can ask similar questions as in the testing problem. (remind the audience the questions)

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Want to minimize:

$$\mathbb{E}\left[(\Theta - \hat{\Theta}_\tau)^2 + c\tau\right]$$

over stopping times.

Markovian embedding

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- Define $\Psi(n, \hat{\Theta}_n) = \mathrm{Var}(\Theta | \mathscr{F}_n^X)$, then V can be written in the θ_0 coordinate:

$$V(n, \theta_0) = \inf_{\tau \in \mathscr{T}} \mathbb{E}_{n,\theta_0}[\Psi(n+\tau, \hat{\Theta}_{n+\tau}) + c\tau].$$

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- Note that $M_n := \Psi(n, \hat{\Theta}_n) + \hat{\Theta}_n^2$ is a martingale, we can further write

$$\begin{aligned} V(n, \theta_0) &= \Psi(n, \theta_0) + \inf_{\tau \in \mathscr{T}} \mathbb{E}_{n,\theta_0} \left[\sum_{i=0}^{\tau} \left(c - \left(\hat{\Theta}_{i+1}^2 - \hat{\Theta}_i^2 \right) \right) \right] \\ &=: \Psi(n, \theta_0) + v(n, \theta_0). \end{aligned}$$

Main result: conditions for space-monotonicity

First-order stochastic dominance

If $\theta_0 \leq \tilde{\theta}_0$, then $\mathbb{P}(\hat{\Theta}_n^{\theta_0} \leq a) \geq \mathbb{P}(\hat{\Theta}_n^{\tilde{\theta}_0} \leq a)$, for all $a \in \mathbb{R}$ and all $n \geq 0$.

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This implies a **one-sided** stopping boundary.

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- What about concavity? Not to be expected.
- What about time-monotonicity? We have some partial results. e.g. . . .

The structure of continuation/stopping

Outline

Dynamic pricing under a binary prior

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Economic & operations research literatures: incomplete learning, myopic strategy . . . . c.f. Mclennan 1984, Harrison 2012.

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- The value can be written as

$$V(\pi) = \sup_{\{p_n\}_{n\geq 0}} \mathbb{E}_\pi \left[\sum_{n=0}^{\infty} e^{-rn} p_n \left(\Pi_n D^1(p_n) + (1-\Pi_n) D^0(p_n) \right) \right].$$

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This leeds to some problems of "exploration-exploitation" type.

Summary

To summarise the talk:

- We study the Bayesian sequential **testing** and **estimation** problems in discrete time.
- The unknown parameter is taken from the **exponential family**.
- The prior can be arbitrary.
- In general, no explicit solutions. We are after structural properties.
- The problems we study open up doors to certain control problems.

Thank you for your attention!

Contact: yuqiong.wang@math.uu.se