Optimal tournament design in continuous time

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Outline

Stopping with incomplete information

- Optimal stopping problems often concern:

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- Combine optimal stopping and filtering theory: optimising while learning.
- Applications: e.g. statistics.

Sequential testing & Optimal stopping

Classical problem: Testing the unknown drift of a BM.

- Observe the trajectory of a BM with unknown drift:

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- The time to stop observing is part of the decision.

Statistics & Optimal stopping

- The minimised cost V:

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$$V(\pi) = \inf_\tau \mathbb{E}_\pi[c\tau + \Pi_\tau \wedge (1-\Pi_\tau)]$$

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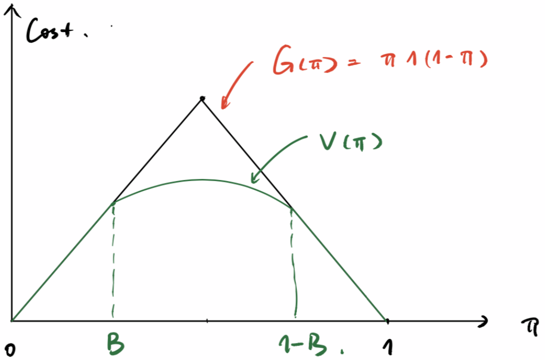
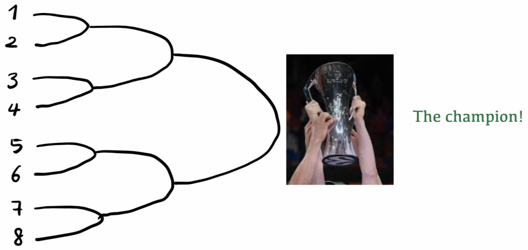
- Standard method: explicit solution: Shiryaev (1969).

Statistics & Optimal stopping



Motivation: knock-out tournament design

- Football, table tennis,. . .



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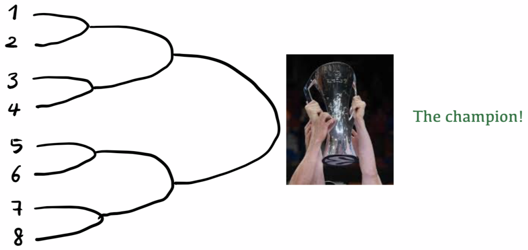
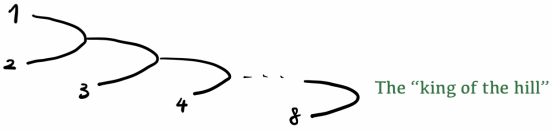


- Children game, sorting:



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We study a continuous-time football-type game with 2^n players.

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Consider a game of 4 players. Want to minimise the probability of "knocking out" the best player, with little observation cost.

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where

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- Initially, we have a uniform prior distribution on

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \dots, \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \right\}, \quad \text{(24 configurations)}$$

The semi-finals

- Define $\Pi_t^{ij} := \mathbb{P}(\Theta^{ij} = \frac{1}{2} | X_t^{ij})$
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- The games in the same round uses the same strategy. i.e.,

$$\tau_b^{12} := \inf\{t : \Pi_t^{12} \notin (b, 1-b)\},$$

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$$\tau_b^{34} := \inf\{t : \Pi_t^{34} \notin (b, 1-b)\}.$$

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- Cost of the semi-finals:

$$\mathbb{E}_{\frac{1}{2}}\left[c\left(\tau_b^{12} + \tau_b^{34}\right)\right] = 2\mathbb{E}_{\frac{1}{2}}\left[c\tau_b^{12}\right].$$

- This cost solves an ODE for a chosen b.

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- Assume that player 1 and player 3 have won their games, i.e.,

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- We consider the final: the "regret" is

$$a = \mathbb{P}(\text{player 2 or 4 is the best} | X_t^{12} = x_{12}, X_t^{34} = x_{34})$$

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- We can show that

$$a = b.$$

The final

- We optimize:

$$\inf_{\tau} \mathbb{E}_{\frac{1}{2}} \left[a + (1-a)\left(\Pi_{\tau}^{13} \wedge \left(1-\Pi_{\tau}^{13}\right) + c\tau\right)\right]$$
$$= b + (1-b) \inf_{\tau} \mathbb{E}_{\frac{1}{2}} \left[\left(\Pi_{\tau}^{13} \wedge \left(1-\Pi_{\tau}^{13}\right) + \frac{c}{1-b}\tau\right)\right]$$
$$= b + (1-b) A(b).$$

- As a solution to the Shiryaev problem,

$$\tau_*^{13} := \inf\{t : \Pi_t^{13} \notin (B, 1-B)\},$$

where $B = B(b)$ and $B \in (0, \frac{1}{2})$.

Optimising the tournament

- Summing up the cost of the semi-final and the final, we seek b to optimise

$$2c\mathbb{E}_{\frac{1}{2}}[\tau_b^{12}] + b + (1-b)A(b).$$

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- with some calculations, it turns out we look for a pair $(b, B) \in (0, \frac{1}{2}) \times (0, \frac{1}{2})$ that

$$\begin{cases} \psi'(b) = \frac{1-B}{4c}, \\ \psi'(B) = \frac{1-b}{2c}. \end{cases}$$

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- ψ being concave implies that

$$0 < B < b < \frac{1}{2}.$$

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- Asymmetric strategies? The whole problem becomes highly asymmetric.
- King of the hill problem?
- Non-knock-out systems: is Monrad system the best?
- Sorting with uncertainty?

Many possibilities. . .

Thank you for your attention!

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