Stopping problems with an unknown state

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Outline

Motivation: a simple hiring game

- Two companies interview a candidate and observe respectively:

$$X_t^1 = \theta t + W_t^1,$$
$$X_t^2 = \theta t + W_t^2,$$

where W^1, W^2 independent.

- θ is the “true” ability level. e.g., $\theta \in \{1, -1\}$, “strong/weak” candidate.
- At any time, the companies can choose to stop the interview process and hire the candidate.
- When hired, the company gain θ.

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- Problem: there is only one candidate - competition in the hiring market.
- Companies must act before their competitor:

$$J_1 = \mathbb{E}[\theta \mathbb{1}_{\tau_1 \leq \tau_2}],$$
$$J_2 = \mathbb{E}[\theta \mathbb{1}_{\tau_2 < \tau_1}].$$

- Companies also observes the **inaction** of the competitor.
- Want to preempty each other, but the inaction of the competitor also affects their belief.

This game is difficult to solve. **However, from a single stopper perspective:**

- Opportunities to stop would disappear (random time horizon).
- The rate of disappearing depends on the state θ.

It motivates us to consider problems with state-dependent random horizon.

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Problem formulation

The ingredients of our problem:

- Consider Bernoulli random variable θ

$$\mathbb{P}_\pi(\theta = 1) = \pi = 1 - \mathbb{P}_\pi(\theta = 0),$$

  and Brownian motion W independent of θ.
- Let random time γ depend on θ, and be independent of W:

$$\mathbb{P}_\pi(\gamma > t | \theta = i) = F_i(t), \quad i = 0, 1,$$

  where F_i continuous, non-increasing, $F_i(0) = 1$.
- Let the underlying X be a diffusion that depend on θ:

$$dX_t = \mu(X_t, \theta)dt + \sigma(X_t)dW_t,$$

  and denote $\mu_i(x) = \mu(x, i), i = 0, 1$.
- Let the payoff $g, h : [0, \infty) \times \mathbb{R} \times \{0, 1\}$ depend on θ, and denote $g_i(t, x) := g(t, x, i)$ and $h_i(t, x) := h(t, x, i)$.

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Problem formulation

- We consider the following problem:

$$V = \sup_{\tau \in \mathscr{T}^{X,\gamma}} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau < \gamma\}} + h(\gamma, X_\gamma, \theta) 1_{\{\tau \geq \gamma\}} \right]. \quad (1)$$

– $\mathscr{F}^{X,\gamma}$: generated by X and $1_{\cdot \geq \gamma}$,
–$\mathscr{T}^{X,\gamma}$: the set of $\mathscr{F}^{X,\gamma}$-stopping time.

- Note that
  - $g(t,x,\theta) = g(t,\theta)$, $h(t,x,\theta) = h(t,\theta)$: statistical problems, X serves as an observation process
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Outline

Incomplete to complete information

- Observe that:

$$\hat{V} = \sup_{\tau \in \mathscr{T}^X} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau < \gamma\}} + h(\gamma, X_\gamma, \theta) 1_{\{\tau \geq \gamma\}} \right] = V. \qquad (2)$$

- Define the conditional probability process:

$$\Pi_t := \mathbb{P}_\pi(\theta = 1 | \mathscr{F}_t^X)$$

Proposition

We have

$$V = \sup_{\tau \in \mathscr{T}^X} \mathbb{E}_\pi \Big[g_0(\tau, X_\tau)(1 - \Pi_\tau) F_0(\tau) + g_1(\tau, X_\tau) \Pi_\tau F_1(\tau) \qquad (3)$$

$$- \int_0^\tau h_0(t, X_t)(1 - \Pi_t) dF_0(t) - \int_0^\tau h_1(t, X_t) \Pi_t dF_1(t) \Big].$$

Moreover, if $\tau \in \mathscr{T}^X$ is optimal in (2), then it is also optimal in (1).

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Moreover, if $\tau \in \mathscr{T}^X$ is optimal in (2)*, then it is also optimal in* (1)*.*

Incomplete to complete information

- The pair (X, Π) satisfies:

$$\begin{cases} dX_t = (\mu_0(X_t) + (\mu_1(X_t) - \mu_0(X_t))\Pi_t)\,dt + \sigma(X_t)d\hat{W}_t \\ d\Pi_t = \omega(X_t)\Pi_t(1-\Pi_t)\,d\hat{W}_t, \end{cases}$$

where $\omega(x) = (\mu_1(x) - \mu_0(x))/\sigma(x)$.

- The process

$$\hat{W}_t := \int_0^t \frac{dX_t}{\sigma(X_s)} - \int_0^t \frac{1}{\sigma(X_t)}(\mu_0(X_s) + (\mu_1(X_s) - \mu_0(X_s))\Pi_s)\,ds$$

is the innovation process (a $\mathbb{P}_\pi$-Brownian motion).

- The process $\Phi := \frac{\Pi_t}{1-\Pi_t}$ satisfies

$$d\Phi_t = \omega(X_t)\Phi_t(\omega(X_t)\Pi_t\,dt + d\hat{W}_t) \qquad (4)$$

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A measure change

Lemma

For any $t \geq 0$, denote by $\mathbb{P}_{\pi,t}$ the measure $\mathbb{P}_\pi$ restricted to $\mathscr{F}_t$, $\pi \in [0,1]$. We then have

$$\frac{d\mathbb{P}_{0,t}}{d\mathbb{P}_{\pi,t}} = \frac{1+\varphi}{1+\Phi_t}.$$

- Under $\mathbb{P}_0$, (X,Φ) satisfies

$$\begin{cases} dX_t = \mu_0(X_t)\,dt + \sigma(X_t)\,dW_t \\ d\Phi_t = \omega(X_t)\Phi_t\,dW_t \end{cases} \tag{5}$$

- Introduce the process

$$\Phi_t^\circ := \frac{F_1(t)}{F_0(t)}\Phi_t, \tag{6}$$

the likelihood process on $\{\gamma > t\}$.

- Φ_t° satisfies

$$d\Phi_t^\circ = \frac{1}{f(t)}\Phi_t^\circ\,df(t) + \omega(X_t)\Phi_t^\circ dW_t, \quad \Phi_0^\circ = \varphi$$

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Theorem

Denote by

$$v = \sup_{\tau \in \mathscr{T}^X} \mathbb{E}_0 \Big[F_0(\tau)(g_0(\tau, X_\tau) + g_1(\tau, X_\tau)\Phi^\circ_\tau) - \int_0^\tau h_0(t, X_t)dF_0(t) - \int_0^\tau \frac{F_0(t)}{F_1(t)} h_1(t, X_t)\Phi^\circ_t dF_1(t) \Big], \quad (7)$$

where (X, Φ°) *is given by* (5) *and* (6)*. Then* $V = v/(1+\varphi)$*, where* $\varphi = \pi/(1-\pi)$*. Moreover, if* $\tau \in \mathscr{T}^X$ *is an optimal stopping in* (7)*, then it is also optimal in the original problem* (1)*.*

The embedding: $v = v(t, x, \varphi)$.

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Outline

Three motivating examples

- We give 3 examples in one-dimension,
- that reduces to problems only $\Phi^{\circ}-$dependent,
- For solvability, assume

$$F_i(t) = \mathbb{P}_\pi(\gamma > t|\theta = i) = e^{-\lambda_i t}, \quad i = 0, 1,$$

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1. The hiring problem with a dumb competitor

- Hire a person, strong/weak:

$$X_t = \mu(\theta)t + \sigma W_t$$

  with $\mu(0) < \mu(1)$.

- Benefit of hiring:

$$g(t,x,\theta) = \begin{cases} -e^{-rt}c & \text{if } \theta = 0 \\ e^{-rt}d & \text{if } \theta = 1 \end{cases}$$

- Survival probabilities: exponential, with $\lambda_0 < \lambda_1$.
- The stopping problem:

$$V = \sup_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[e^{-r\tau} \left(d1_{\{\theta=1\}} - c1_{\{\theta=0\}} \right) 1_{\{\tau<\gamma\}} \right].$$

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2. Closing a short position under recall risk

- Consider a short position in

$$dX_t = \mu(\theta)X_t dt + \sigma X_t dW_t$$

with $\mu(0) < \mu(1)$.

- The random horizon corresponds to a time when the position is recalled: $\lambda_0 > 0 = \lambda_1$.
- The payoffs are $g(t,x,\theta) = h(t,x,\theta) = xe^{-rt}$, and

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Thank you!

Our paper: Ekström and Wang, *"Stopping problems with an unknown state"*. J. Appl. Probab (2024)