Dynkin ghost games with asymmetry and consolation

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Outline

Let us first play a game

- Scenario 1: No competition.
- Scenario 2: Certain competition.
- Scenario 3: Uncertain competition.

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Two motivating examples

- **Auction:**
  – You don't know if the competitor exists, unless they act.
  – Preemption type.
- **Possible investment opportunities:**
  – Second one to stop gets a slightly worse contract.
  – Different costs for different companies.

This motivates us to study "ghost" games with with asymmetry and consolation.

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Playing a game with a ghost

- Consider a two-player non-zero-sum Dynkin game.
- Player 1 and 2 observes the same process X (continuous, strong Markov), and each chooses a time to stop: γ_1, γ_2.
- **Key feature 1: Uncertain competition.** Each player is uncertain about the existence of a competitor.

$$\theta_i = \text{"Player i has competition"} \in \{0,1\}, \text{ for } i \in \{1,2\}.$$

- We define

$$\hat{\gamma}_{3-i} := \begin{cases} \gamma_{3-i} & \text{on } \{\theta_i = 1\} \\ \infty & \text{on } \{\theta_i = 0\}. \end{cases}$$

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- **Key feature 3: Consolation.** Player i gets $h_i(X_{\tau_i})$ when they stop, if they were the second to stop.
- Preemption type: $g_i \geq h_i, i \in \{1,2\}$.
- The expected discounted payoffs are defined as

$$J_1(x;\gamma_1,\gamma_2) := \mathbb{E}_x[e^{-r\gamma_1} g_1(X_{\gamma_1})\mathbb{1}_{\{\gamma_1<\hat{\gamma}_2\}} + e^{-r\gamma_2} V^{h_1}(X_{\gamma_2})\mathbb{1}_{\{\gamma_1\geq\hat{\gamma}_2\}}],$$

and

$$J_2(x;\gamma_1,\gamma_2) := \mathbb{E}_x[e^{-r\gamma_2} g_2(X_{\gamma_2})\mathbb{1}_{\{\gamma_2\leq\hat{\gamma}_1\}} + e^{-r\gamma_1} V^{h_2}(X_{\gamma_1})\mathbb{1}_{\{\hat{\gamma}_1<\gamma_2\}}].$$

where V^f represents the “American value” with payoff f.

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- At the beginning of the game, each player estimates their probability of competition:

$$\mathbb{P}(\theta_i = 1) = p_i.$$

Then they adjust their belief processes $\Pi^i_t = \mathbb{P}(\theta_i = 1 | \mathscr{F}^X_t, \hat{\gamma}_{3-i} > t)$ by observing:

  - the underlying X,
  - the lack of action of their competitor.
- Note that we can "fool" our competitor, a pure-strategy equilibrium wouldn't exist!
- This means $\gamma' s$ should be randomised stopping times:

$$\gamma_1 = \inf\{t \geq 0 : \Gamma^1_t \geq U_1\}$$

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where $U_1, U_2 \sim Unif(0,1)$, independent.
- Γ^1, Γ^2 are $[0,1]$-valued $\mathscr{F}-$adapted controls, right continuous, non-decreasing, and $\Gamma^i_{0-} = 0$.

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Ghost games with preemption

- Furthermore, Π_t^i is a function of Γ_t^{3-i}:

$$\Pi_t^i = \begin{cases} \frac{p_i(1-\Gamma_t^{3-i})}{1-p_i\Gamma_t^{3-i}} & \text{if } p_i < 1 \\ 1 & \text{if } p_i = 1 \end{cases}$$

- We seek for conditions such that a Nash equilibrium (Γ_1^*, Γ_2^*) exists:

$$J_1(x; \Gamma_1, \Gamma_2^*) \leq J_1(x; \Gamma_1^*, \Gamma_2^*) \text{ and } J_2(x; \Gamma_1^*, \Gamma_2) \leq J_2(x; \Gamma_1^*, \Gamma_2^*).$$

- And we are interested in the associated values:

$$u_1(x, p_1, p_2) = J_1(x; \Gamma_1^*, \Gamma_2^*) \quad \& \quad u_2(x, p_1, p_2) = J_2(x; \Gamma_1^*, \Gamma_2^*).$$

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Our main result

A verification theorem

Let two continuous functions $u_1, u_2 : \mathbb{I}_{\{\times\}}[0,1]^2 \to [0,\infty)$ and a pair (Γ^1, Γ^2) be given. Assume that $u_i \le V^{g_i}$, and that $\Gamma^i_{\tau_{g_i}} = 1$, $i = 1,2$. Define on $[0,\infty)$ two processes

$$M^1_t := e^{-rt}(1 - p_1\Gamma^2_t)u_1(X_t, \Pi^1_t, \Pi^2_t) + p_1 \int_{[0,t]} e^{-rs} V^{h_1}(X_s)\, d\Gamma^2_s$$

and

$$M^2_t := e^{-rt}(1 - p_2\Gamma^1_{t-})u_2(X_t, \Pi^1_{t-}, \Pi^2_{t-}) + p_2 \int_{[0,t)} e^{-rs} V^{h_2}(X_s)\, d\Gamma^1_s,$$

and assume that

(i) M^i is a supermartingale, and it is a martingale on $[0, \gamma_i(u)]$ for any $u < 1$, $i = 1,2$;

(i') M^2 is continuous and only has downward jumps;

(ii) $u_1(X_t, \Pi^1_t, \Pi^2_t) \ge g_1(X_t)$ and $u_2(X_t, \Pi^1_{t-}, \Pi^2_{t-}) \ge g_2(X_t)$ for all $t \ge 0$ $\mathbb{P}_x$-a.s.;

(iii) $\Gamma^1_t = \int_{[0,t]} 1_{\{u_1(X_s, \Pi^1_s, \Pi^2_s) = g_1(X_s)\}}\, d\Gamma^1_s$ and $\Gamma^2_t = \int_{[0,t]} 1_{\{u_2(X_s, \Pi^1_{s-}, \Pi^2_{s-}) = g_2(X_s)\}}\, d\Gamma^2_s$.

Then (Γ^1, Γ^2) is a Nash equilibrium.

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Case 1: symmetric cases with martingale consolation

Conclusion: the solutions can be constructed explicitly.

- Assumption: $e^{-rt\wedge\tau_g} V^h(X_{t\wedge\tau_g})$ is a martingale.
- e.g., $\{x : V^g(x) > g(x)\} \subseteq \{x : V^h(x) > h(x)\}$,
- Guess: by the indifference principle, the value of Player 1 should be

$$u_1(x, p_1) := (1 - p_1)V^g(x) + p_1 V^h(x).$$

- (Γ^1, Γ^2) can be constructed accordingly, and verified.

Remark

- *The equilibrium values only depend on p_1.*
- *The (X, Π_1) process is reflected along the stopping boundary towards the continuation region.*

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- e.g., $\{x : V^g(x) > g(x)\} \subseteq \{x : V^h(x) > h(x)\}$,
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Remark

- *The equilibrium values only depend on p_1.*
- *The (X,Π_1) process is reflected along the stopping boundary towards the continuation region.*

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special case: $h = 0$. Studied by **De Anglis and Ekström (2020)**. This is where we generalise from.

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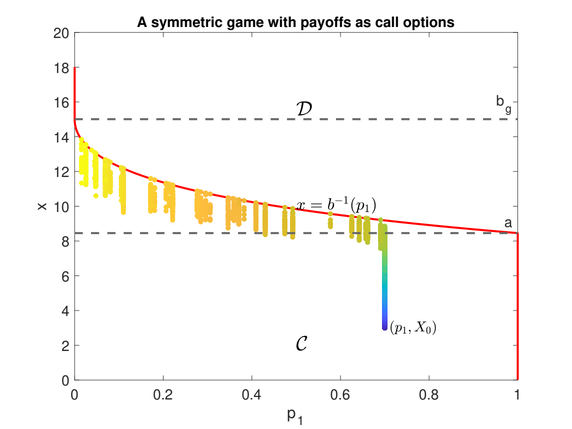
Example 2: Call option payoff and consolation

let X be a GBM, $dX_t = \mu X_t dt + \sigma X_t dW_t$. $g(x) = (x-K)^+$ and $h(x) = (x-L)^+$, for positive constants $K < L$.



In this illustration, $g(x) = (x-3)^+$, $h(x) = (x-4)^+$. $\mu = 0.08$, $\sigma = 0.01$ and $r = 0.1$.

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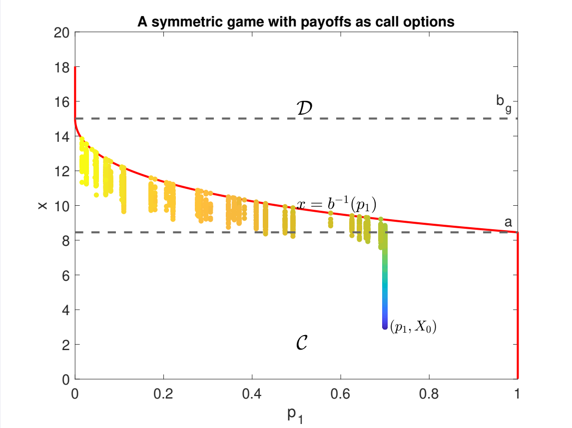

In this illustration, $g(x) = (x-3)^+$, $h(x) = (x-4)^+$. $\mu = 0.08$, $\sigma = 0.01$ and $r = 0.1$.

Case 2: symmetric cases with general consolation

Conclusion: Sometimes we can solve an ODE.

- Candidate value: $u_1(x,p_1,p_2) := (1-p_1)V^g(x) + p_1\mathbb{E}_x[e^{-r\gamma_2}V^h(X_{\gamma_2})]$.
- We can still argue that $u_1(x,p_1,p_2) = u_1(x,p_1)$.
- On the stopping boundary, it holds that

$$(1-p_1)\frac{\partial u_1}{\partial p_1} + u_1 = V^h$$

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- Let the stopping boundary be $p_1 = b(x)$ and make the ansatz

$$u_1(x, p_1) = c(p_1)\psi(x)$$

- We get the system

$$\begin{cases} c(b(x))\psi(x) = g(x) \\ (1 - b(x))c'(b(x))\psi(x) + c(b(x))\psi(x) = V^h(x). \end{cases}$$

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Cases with two sided payoffs: similar.

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Conclusion: Sometimes we can guess and verify, in general it breaks down.

Example 4: stopping in the same direction

- Consider a case where the investors have individual investment costs, $g_i(x) = (x - K_i)^+$ and $h_i(x) = (x - L_i)^+$, with $K_i \leq L_i$.
- We assume $p_1 \leq p_2$, and $K_2 \leq K_1$.
- Similar intuition carries here: Player 1 should stop later.
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$$u_1(x, p_1) := (1 - p_1) V^g(x) + p_1 V^h(x).$$

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- Assume there is no consolation: $h_1 = h_2 = 0$.
- We expect a lower stopping surface lower boundary surface $\{x = L(p_1, p_2)\}$ and an upper boundary surface $\{x = U(p_1, p_2)\}$.
- Player 1 exercises with some intensity only on L, and Player 2 only on U.
- The candidate values are given by

$$\begin{cases} u_1(x, p_1, p_2) = C_1(p_1, p_2)\psi(x) + D_1(p_1, p_2)\phi(x), \\ u_2(x, p_1, p_2) = C_2(p_1, p_2)\psi(x) + D_2(p_1, p_2)\phi(x), \end{cases}$$

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Summary

- We study a Dynkin game with uncertain competition.
- In the presence of asymmetry and consolation, in general, we don't have explicit solutions.
- In some cases, we can hope for explicit solutions for one of the players.

Future work

- The stopping boundary is in general a surface $f(x, p_1, p_2) = 0$.
- A variational inequality,solvability?
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Our paper: Ekström and Wang, *"Dynkin ghost games with asymmetry and consolation "*. arXiv:2411.04802