On hypoellipticity of degenerate operators in testing and detection problems

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Outline

- A stopper observes a 1D Brownian motion with drift

$$dX_t = \theta t + dW_t, \quad X_0 = 0.$$

with $P(\theta = 1) = 1 - \theta = 0 = \pi \in [0, 1]$.

- Want to test hypotheses for its drift, e.g., $H_0 : \theta = 0$ vs $H_1 : \theta = 1$.
- We are penalized for making a mistake, and have a constant observation cost c per unit time.

$$V = \inf_{(\tau, d)} \mathbb{E}[\mathbb{P}(d \neq \theta) + c\tau].$$

- **"Sequential testing problem"**, can be formulated as an optimal stopping problem in the posterior probability process, as

$$V(\pi) = \inf_{\tau} \mathbb{E}[\Pi_\tau \wedge (1 - \Pi_\tau) + c\tau]$$

where $\Pi_t := \mathbb{P}(\theta = 1 | \mathcal{F}_t^X)$.

- This 1D problem can be formulated as a free-boundary problem and solved explicitly [cf. Shiryaev (1978)].

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A challenge when in higher dimensions: degeneracy

Multi-dimension setting? many possibilities...

Let's for example, look at the following problem:

- Consider now the Brownian motion has 3 possible drifts instead of 2: $P(\theta = i) = \pi_i, i \in \{0, 1, 2\}$.
- The sufficient statistics in this case is (Π^1, Π^2).
- There are two underlying coordinates but only one underlying Brownian source. The operator is degenerate elliptic.
- Can be resolved: Π_t^i are functions of (t, X_t). Can formulate it in the $(t, x)-$coordinate: uniformly parabolic [cf. Zhitlukhin and Shiryaev (2011)].

A challenge when in higher dimensions: degeneracy

What about the following cases?

- When θ can change its value at exponential times. e.g. **classic quickest detection**:

$$dX_t = 1_{t \geq \theta} dt + dW_t.$$

  with $\mathbb{P}(\theta = 0) = \pi$ and $\mathbb{P}(\theta > t | \theta > 0) = e^{-\lambda t}$.
- i.e., θ changes from 0 to 1 then never changes back.
- Want to declare the change point asap without a false alarm:

$$V = \inf_{\tau} \{ \mathbb{P}(\tau < \theta) + \mathbb{E}[(\tau - \theta)^+] \}$$

- Problem: the posterior process $\Pi_t := \mathbb{P}(\theta \leq t | \mathcal{F}_t)$ depends on the whole path. No longer possible to formulate it in (t, x).
- Can solve in $1D$, can be degenerate in Π when extended to higher dimensions. e.g., θ changes to two possible values.

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Beyond testing and detection: partial information

- When the problem is $X-$dependent. More applications.
- e.g., a "hiring problem" application:

$$X_t = \theta t + W_t,$$

and the payoff upon stopping at τ being

$$e^{-r\tau} X_\tau .$$

After filtering, the problem has (X, Π) as its state.

These motivate us to study properties of these degenerate cases.

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Problem formulation

The ingredients of our problem:

- A continuous time Markov chain $(\theta_t)_{t\geq 0}$ taking values in $\bar{n} := \{0, 1, \ldots, n\}$ with generator $Q = (q_{ij})_{i,j\in\bar{n}}$,
- $k-$dimensional Brownian motion $W = (W^1, \ldots, W^k)$ independent of θ.
- $k < n$.

We refer to

- $Q \equiv 0$: the "testing case",
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Problem formulation

We consider

$$dX_t = \sum_{j=0}^{n} 1_{\theta_t = j} \lambda_j dt + dW_t, \quad X_0 = 0. \tag{1}$$

and stopping problems of the form

$$V = \sup_{\tau \in \mathcal{T}} \mathbb{E}_\pi \left[e^{-\int_0^\tau r(\Pi_s)ds} g(\Pi_\tau) + \int_0^\tau e^{-\int_0^t r(\Pi_s)ds} h(\Pi_t) dt \right]. \tag{2}$$

- Here g, h, r are continuous, $r \geq 0$, $\lambda_i \in \mathbb{R}^k$, $i \in \{0, 1, \ldots, n\}$.
- The posterior Π lives on the n-dimensional simplex P_{n+1} with

$$\Pi_t^i = \mathbb{P}_\pi(\theta_t = i \mid \mathcal{F}_t^X) \quad \text{for } i \in \{0, \ldots n\}$$

Problem formulation

- By standard filtering theory, Π_t has dynamics:

$$d\Pi_t^j = \underbrace{\sum_{i=0}^{n} q_{ij}\Pi_t^i dt}_{\text{Drift from } Q} + \underbrace{\Pi_t^j(\lambda_j - \overline{\lambda}_t) \cdot d\tilde{W}_t}_{\text{Diffusion from } W_t}$$

- where $\overline{\lambda}_t = \sum_{i=0}^{n} \lambda_i \Pi_t^i$, and $\tilde{W}_t$ is the innovation process.
- The stopping problem is governed by the infinitesimal generator $\mathcal{L}_\pi$ for this Π_t process

$$\mathcal{L}_\pi = \underbrace{\frac{1}{2}\sum_{i,j=0}^{n} \pi_i \pi_j (\lambda_i - \overline{\lambda}) \cdot (\lambda_j - \overline{\lambda}) \frac{\partial^2}{\partial \pi_i \partial \pi_j}}_{\text{Diffusion (degenerate)}} + \underbrace{\sum_{i,j=0}^{n} q_{ij}\pi_i \frac{\partial}{\partial \pi_j}}_{\text{Drift (from } Q)}$$

- **It degenerates everywhere.**

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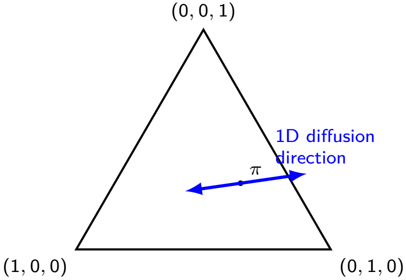
Geometric intuition of degeneracy ($n = 2, k = 1$)

- Π_t lives on the $n-$dim simplex.
- Observations are driven by a k–dimensional Brownian motion.
- **Local picture:** at each interior point π, randomness initially acts only in a k–dimensional subspace of the n–dimensional tangent space.



Hypoellipticity and the Hörmander's condition

- **Question:** Even if the operator is not elliptic, can we still recover regularity (e.g., smoothness) of the value function?
- **Our hope:** Hypoellipticity (the property that u smooth if $\mathcal{L}u$ smooth).
- **Intuition:** The operator may be degenerate, but the randomness "spreads" through the system.
- The "missing directions" from the k-dimensional diffusion might be restored via **iterated Lie brackets**.

Hörmander (1967)

Write $\mathcal{L}_\pi = \sum_{r=1}^{k} D_r^2 + D_0$, where D_i's are C^∞ vector fields. If

$$Lie(D_0, D_1, \ldots, D_K)$$

spans the tangent space of the simplex at every point in $int(P_{n+1})$, the Hörmander's condition is satisfied, and the operator $\mathcal{L}_\pi$ is hypoelliptic.

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Existing work

- [cf. Caffarelli and Friedman (1981)] studied the problem where $n = k$, with

$$g(\pi) = a_0(1-\pi_0) \wedge \cdots \wedge a_n(1-\pi_n) \quad h(\pi) = \sum_{i=0}^{n} c_i \pi_i.$$

- They commented on the case where $k < n$ and gave the 1D, 3 drift example.
- Few literature in the filtering field: [cf. Peskir (2022), Ernst et al (2022)]

Our goal: characterize when the Hörmander's condition holds for $\mathcal{L}_\pi$.

A better coordinate: Φ_t

- We do a change of coordinate to the **posterior likelihood process** Φ_t:

$$\Phi_t^i = \frac{\Pi_t^i}{\Pi_t^0} \quad \text{for } i = 1, \ldots, n \quad (\text{Note: } \Phi_t^0 \equiv 1)$$

- Why Φ?
- This map is a C^∞-diffeomorphism from $int(P_{n+1}) \to (0, \infty)^n$. **Hypoellipticity is preserved**.
- Define $a_i \in \mathbb{R}^k$ for $i = 1, \ldots, n$: $a_i := \lambda_i - \lambda_0$, and $\Sigma_{ij} = a_i \cdot a_j$,

$$d\Phi_t^i = \left(\sum_{m=0}^{n} \Phi_t^m (q_{mi} - q_{m0}\Phi_t^i) + \frac{1}{Y} \sum_{m=1}^{n} \Sigma_{i,m} \Phi_t^i \Phi_t^m \right) dt + \Phi_t^i a_i \cdot d\tilde{W}_t \quad (3)$$

with $\Phi_0^i = \phi_i$ and $Y_t := \sum_{i=0}^{n} \Phi_t$.

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Sum-of-squares decomposition

Denote by $y(\phi) = \sum_{i=0}^{n} \phi_i$, the generator $\mathcal{L}$ for the Φ process is:

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} \Sigma_{ij}\phi_i\phi_j \frac{\partial^2}{\partial\phi_i\partial\phi_j} + \frac{1}{y(\phi)}\sum_{i,j=1}^{n} \Sigma_{ij}\phi_i\phi_j \frac{\partial}{\partial\phi_j} + \sum_{j=1}^{n}\sum_{i=0}^{n}(q_{ij} - q_{i0}\phi_j)\phi_i \frac{\partial}{\partial\phi_j}$$

We can write $\mathcal{L} = D_0^J + \frac{1}{2}\sum_{r=1}^{k} D_r^2$ with

- **Diffusion fields.**

$$D_r := \sum_{i=1}^{n} a_{ir}\,\phi_i\,\partial_{\phi_i}, \qquad \text{where } a_i = (a_{i1}, \ldots, a_{ik}).$$

- **Drift and switching fields.**

$$D_0^J := \frac{1}{y(\phi)}\sum_{i,j=1}^{n} \Sigma_{ij}\phi_i\phi_j\,\partial_{\phi_j} - \frac{1}{2}\sum_{i=1}^{n} \|a_i\|^2\phi_i\,\partial_{\phi_i} + \sum_{j=1}^{n}\sum_{i=0}^{n}(q_{ij} - q_{i0}\phi_j)\,\phi_i\,\partial_{\phi_j},$$

Outline

Main Results: characterization in the testing case (Q=0)

We first write $\mathcal{L}$ in the "sum of squares" form $\mathcal{L} = D_0 + \frac{1}{2}\sum_{r=1}^{k} D_r^2$.

- **Key observations:**

  1. The diffusion fields commute:

$$[D_r, D_u] = 0 \quad \text{for all } r, u \in \{1, \ldots, k\}$$

  2. The bracket of the drift and a diffusion field **stays in the diffusion span**:

$$[D_0, D_u] = \sum_{s=1}^{k} c_s(\phi) D_s \in \text{span}\{D_1, \ldots, D_k\}$$

Theorem 0

Let $Q = 0$. Let $A = (a_1, \ldots, a_n) \in \mathbb{R}^{k \times n}$.
If $\mathbf{n} > \mathbf{k+1}$, the Hömander's condition FAILS.
If $\mathbf{n} = \mathbf{k+1}$, the Hömander's condition HOLDS if and only if $rank(A) = k$ and the vector $(\|a_1\|^2, \ldots, \|a_n\|^2)$ is not in the rowspace of A.

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Sufficient conditions for the detection case ($Q \neq 0$)

- Add the "switch" field $J := \sum_{j=1}^{n} \sum_{i=0}^{n} (q_{i,j} - q_{i,0}\phi_j)\phi_i \partial_{\phi_j}$
- **The Key:** The closure mechanism is broken. The Lie bracket of J with the diffusion fields D_r creates *new* vector fields.

$$[J, D_r], \quad [D_s, [J, D_r]], \quad \text{etc.}$$

- First-level brackets:

$$[J, D_r] \quad \text{produce new diagonal-type fields with coefficients involving } (q_{mi})_{m \neq i}.$$

- Iterating:

$$[D_s, [J, D_r]], \quad [D_{s_2}, [D_{s_1}, [J, D_r]]], \ \ldots$$

creates a family of polynomial-weighted diagonal fields.

Sufficient conditions for the detection case ($Q \neq 0$)

Theorem 1 (Sufficient Condition 1)

The Hömander's condition holds if: (1) The drift-difference vectors $a_1, ..., a_n$ are pairwise distinct, and (2) **For each** coordinate $i \in \{1, ..., n\}$, there exists some state $m \neq i$, $m \in \{0, ..., n\}$ such that $q_{mi} > 0$.

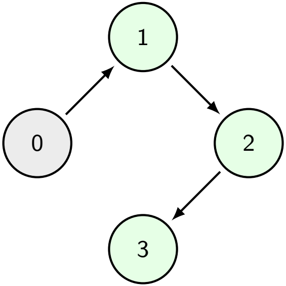
Remark:

- Proof: by construction.
- Intuition: incoming information for every hypothesis.
- (2) is much weaker than Q being irreducible.

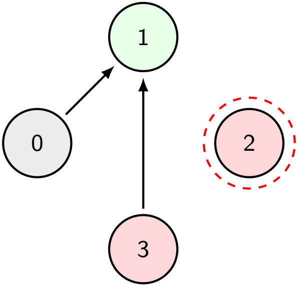
Global v.s. Local

Thm 1 condition (2): some examples

Condition (2) holds

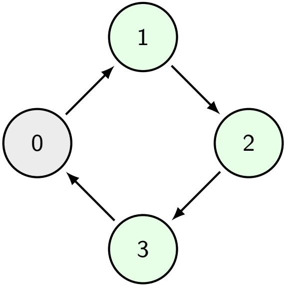


Condition (2) fails

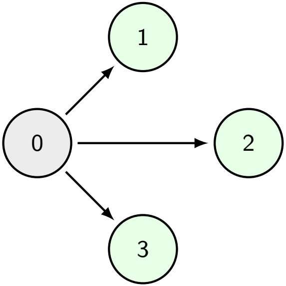


Irreducibility vs Thm 1 condition (2)

Q irreducible (strongly connected)



Q not irreducible, but condition (2) holds



Sufficient conditions for the detection case ($Q \neq 0$)

If Theorem 1 fails?

- If only one column i such that $q_{0i} = 0$ and $q_{mi} = 0$ for all m. If there exits some p such that $q_{p0} > 0$, can use the same construction. Hypoellipticity holds.
- If there are strictly more than one such columns, this construction fails.
- But can span a smaller space.

Sufficient conditions for the detection case ($Q \neq 0$)

Theorem 2 (Sufficient Condition 2)

Assume $q_{mj} = 0$ for all $m \neq j$ and that there is at least one j such that $q_{j0} > 0$ (at least one state can jump back to 0).
Define the $n \times (k+2)$ augmented matrix $\tilde{A}$:

$$\tilde{A} = \begin{bmatrix} a_{11} & \dots & a_{k1} & \|a_1\|^2 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{1n} & \dots & a_{kn} & \|a_n\|^2 & 1 \end{bmatrix}$$

Then, dim $Lie(D_0^J, ..., D_k) = \min(rank(\tilde{A}), n)$. Hypoellipticity holds if $rank(\tilde{A}) = n$, which can **never** hold if $\mathbf{n} > \mathbf{k+2}$.

Consequences of Hörmander

Parabolic Hörmander's condition is immediate:

- Define $\bar{D}_0 := -\partial_t + D_0$, dim Lie$\{D_0^J, D_1, D_2, \ldots, D_k\} = n$, then dim Lie$\{\bar{D}_0^J, D_1, D_2, \ldots, D_k\} = n + 1$.
- It follows that the process $(\Phi_t)_{t\geq 0}$ is strong Feller.
- This allows us to deal with $t-$dependent problems.

Hörmander holds in the (ϕ, x) coordinate:

- The operator $\mathcal{L}_{\phi,x}$ for $(\phi, x)-$dependent problems is hypoelliptic on $(0,\infty)^n \times \mathbb{R}^k$ if and only if the operator $\mathcal{L}$ is hypoelliptic on $(0,\infty)^n$.
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Consequences of Hörmander

V is continuous and is the unique viscosity solution of (4) in $\mathring{P}_{n+1}$.

$$\min\{ru - \mathcal{L}_\pi u - h, u - g\} = 0. \tag{4}$$

Define the continuation region

$$\mathcal{C} := \{\pi \in \mathring{P}_{n+1} : V(\pi) > g(\pi)\}$$

and the stopping region

$$\mathcal{D} := \{\pi \in \mathring{P}_{n+1} : V(\pi) = g(\pi)\}.$$

- The VI, continuation and stopping region are now only defined in $\mathring{P}_{n+1}$.
- The boundary is non-attainable: when staring from the interior, Π stays in the interior a.s.
- $\pi \in \partial P_{n+1}$: reduces to lower dimension, or extended as a limit.

Value function regularity in $\mathcal{C}$

We consider only in the **continuation region** $\mathcal{C}$ (where $V > g$).

- If the Hörmander condition holds, and the running payoff $h \in C^\infty(\mathcal{C})$, then the value function V is also $C^\infty(\mathcal{C})$.
- If $r, h \in C^{0,\alpha}(\mathcal{C})$ for $\alpha \in (0,1)$, then the value function $V \in C^{2,\alpha}(\mathcal{C})$.

But no "smooth fit" implied:

- All these regularity results (C^∞ or $C^{2,\alpha}$) hold **only** in the $\mathcal{C}$.
- In particular, hypoellipticity alone **does not imply** the "smooth fit" condition.
- When do we have global C^1? Need boundary points to be probabilistically regular.

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Outline

Example 1: Detecting the change time in multiple coordinate

- We observe an N-dim BM. At time η, K out of N coordinates gain a drift μ.
- There are N Brownian coordinates and $\binom{N}{K}$ possible drifts.
- The generator Q has a specific structure: only the first row is non-zero.

$$Q = \begin{bmatrix} -\binom{N}{K}\lambda & \lambda & \dots & \lambda \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

1. The drift-difference vectors a_j are all distinct. ✓
2. All $q_{0i} > 0$ ✓ .

- The operator **is hypoelliptic** by Theorem 1.

[cf. Ernst et al (2022)]

Example 2: Signal tracking with regime switching

- We observe $X_t = \sum_{j=0}^{n} 1_{\theta_t=j}\lambda_j t + W_t$. with

$$Q = \begin{bmatrix} -\sum_{i=1}^{n} q_i & q_1 & q_2 & \dots & q_n \\ p_1 & -p_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_k & 0 & 0 & \dots & -p_k \end{bmatrix} \tag{5}$$

  with $p_i, q_j > 0$ for all $i \in \{1, k\}, j \in \{1, n\}$.
- Application-wise: monitoring signals from a radar with different levels of disorder. We can consider,e.g.,

$$\inf_d \mathbb{E}[\int_0^T e^{-rt} c(d_t, \theta_t) dt]$$

  with

$$c(d, \theta) = 1_{d=\theta} \sum_{i=1}^{n} c^1(\theta) + 1_{d\neq\theta} \sum_{i=1}^{n} c^2(\theta).$$

- The operator **is hypoelliptic** by Theorem 1 (same as Example 1).

Example 3: Byzantine detection

A detection problem with possible corrupted sensor:

- We observe two processes X^1, X^2:

$$X_t^1 = \mu_0 1_{\eta > t} + \mu_1 1_{\eta \le t} + W_t^1,$$
$$X_t^2 = m_0 1_{\eta > t} + m_1 1_{\eta \le t} + W_t^2,$$

  with $\mathbb{P}_0(\eta = 0) = p, \quad \mathbb{P}_0(\eta > t) = (1-p)e^{-rt}, \quad t \ge 0.$

- The unknown state θ has 4 possiblilities:

$$\theta_0 = \begin{cases} (\mu_1, m_1), & \text{both channels are affected at } t = 0 \\ (\mu_1, m_0), & X^1 \text{ is affected at time } t = 0, \\ (\mu_0, m_1), & X^2 \text{ is affected at time } t = 0, \\ (\mu_0, m_0), & \text{no channels are affected at time } t = 0, \end{cases} \tag{6}$$

We call it Byzantine because of the “Byzantine general problem”.

Example 3: Byzantine detection

- The drift matrix

$$A = -\begin{bmatrix} \mu_1 - \mu_0 & 0 & \mu_1 - \mu_0 \\ 0 & m_1 - m_0 & m_1 - m_0 \end{bmatrix}$$

has full rank (2), but the vector $(\|a_1\|^2, \|a_2\|^2, \|a_3\|^2)$ is always in the row-space of A.

- The generator matrix Q as

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda & -\lambda & 0 & 0 \\ \lambda & 0 & -\lambda & 0 \\ \lambda & 0 & 0 & -\lambda \end{bmatrix}.$$

Theorem 3 applies, the augmented matrix

$$\tilde{A} = -\begin{bmatrix} \mu_1 - \mu_0 & 0 & (\mu_1 - \mu_0)^2 & 1 \\ 0 & m_1 - m_0 & (m_1 - m_0)^2 & 1 \\ \mu_1 - \mu_0 & m_1 - m_0 & (\mu_1 - \mu_0)^2 + (m_1 - m_0)^2 & 1 \end{bmatrix}$$

has rank 3, Hörmander condition holds. In working progress.

Thank you!