Optimal match length in knock-out tournaments

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Outline

Motivation: knock-out tournament design

The goal of a tournament is to single out the best player.

- Some sports play deterministic rules. e.g., football, basketball.
- And some sports play sequential rules. e.g., table tennis, boxing.

In these games sometimes we think later stage games are more important:

- **Table tennis/badminton.** Early rounds are often best of 3 sets, while semifinals/finals are 5 or 7 sets.
- **Boxing.** Title fights are scheduled for 12 rounds, while non titled 4-10 rounds

Does this make sense intuitively? The final determines the champion, while earlier rounds are numerous and each is less "informative" for the final winner.

Does it make sense statistically?

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- In a knock-out tournament, the strongest player must survive **every round**. A single noisy bad game eliminates them permanently.
- Longer matches reduce randomness, but early-round length is expensive because there are usually many early matches.
- In this talk we study how to optimally allocate match lengths in a 2^n-player knock-out tournament.
- We fix a target tournament accuracy

$$\mathbb{P}(\text{best player wins the tournament}) \geq \beta.$$

Two design questions:

1. **Sequential vs. fixed length:** How much can we reduce the *average time per game* by using sequential stopping rules instead of fixed-length matches?
2. **Round allocation:** Should later stage games be longer? If yes, what is the *optimal round-dependent schedule*? Are the different optimal designs for sequential and fixed-length tournaments?

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How much are we better off if we play sequential?

- The behavior of the players are often associated with some uncertainty.
- When a match is played, we can observe its "behavior" as a continuous process.
- We model the match behavior of Player 1 and 2 as a Brownian motion with drift:

$$X_t = \theta t + W_t,$$

where the unknown drift is unobservable:

$$\theta = \begin{cases} 1, & \text{if Player 1 is better than Player 2,} \\ -1, & \text{if Player 2 is better than Player 1,} \end{cases}$$

- We thus test against the hypotheses H_+ and H_-, where

$$H_+ : \theta = 1, \quad H_- : \theta = -1.$$

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- In a **fixed-length** match we observe X until a predetermined time T and declares the winner by the sign of X_T.
- The probability that the strong player is declared the winner is

$$\mathbb{P}(X_T \geq 0 | \theta = 1) = \mathbb{P}(T + W_T \geq 0) = \Phi(\sqrt{T}).$$

- We ask $\Phi(\sqrt{T}) \geq \beta$.
- Thus the minimum deterministic time needed for accuracy β is

$$T = (\Phi^{-1}(\beta))^2, \quad \beta \in (\frac{1}{2}, 1).$$

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- Define the posterior probability process

$$\Pi_t := \mathbb{P}(\theta = 1 | \mathcal{F}_t^X),$$

- The process is observed until $\tau := \inf\{t \geq 0 : \Pi_t \notin (b, 1-b)\}$ for some $b \in (0, \frac{1}{2})$.
- Declare $\theta = 1$ if $\Pi_\tau = 1 - b$ and $\theta = -1$ otherwise. We can show that $\mathbb{P}(\theta = d) = 1 - b$. We thus let $b = 1 - \beta$.
- The expected value of observation time $\mathbb{E}[\tau]$ is characterized through finding the unique solution of the boundary value problem

$$\begin{cases} \frac{1}{2}\pi^2(1-\pi)^2 u_{\pi\pi} + 1 = 0, & \pi \in (b, 1-b), \\ u(b) = u(1-b) = 0, \end{cases}$$

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Theorem 1 (Single match relative efficiency. Doerks–Ekström–Wang ('26))

The function R is strictly decreasing on $(1/2, 1)$, with

$$R(1/2+) = \frac{2}{\pi}, \qquad R(1-) = \frac{1}{4}.$$

Therefore the average time reduction $1 - R(\beta)$ increases with the required accuracy β.

Remark. The minimal reduction is 36% and the maximal reduction is 75%. $f(0+) = \frac{1}{4}$ is well known [cf. Aivazjan('59), Shiyaev('78)], and numerically [cf. Eisenberg('91)] that convergence to 75% is rather slow, and that typical reductions range between 50% and 60%. The monotonicity is new.

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The relative efficiency $R(\beta)$

β	$R(\beta)$	Average reduction
0.80	0.5871	41.29%
0.90	0.5351	46.49%
0.95	0.4897	51.03%
0.99	0.4160	58.40%
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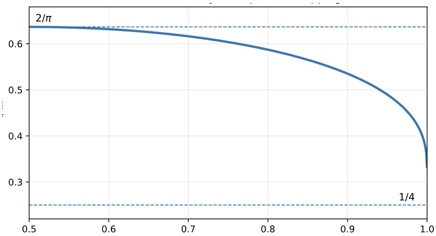

Remark.

- R approaches its limit at $\beta = \frac{1}{2}$ quadratically and $\beta = 1$ logarithmically.
- To reach a 70%, 72%, 74% efficiency reduction, we need error probabilities β to be $2 \times 10^{-7}, 2 \times 10^{-12}, 2 \times 10^{-40}$, respectively.

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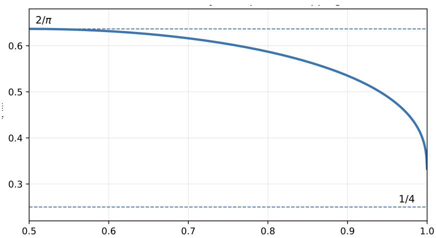

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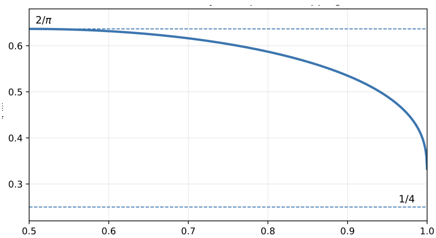

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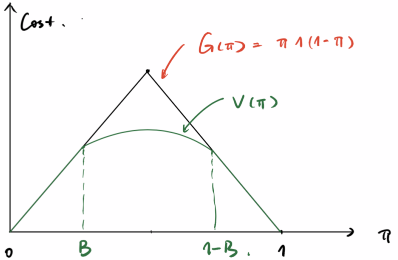
Remark 1. equivalency to the Shiryaev's testing problem

Recall the Bayesian formulation with constant cost per unit time:

$$V = \inf_{\tau,d} \{\mathbb{P}(d=0,\Theta=1) + \mathbb{P}(d=1,\Theta=0) + c\mathbb{E}[\tau]\}. \quad (1)$$

Problem (1) can be written as

$$V(\pi) = \inf_{\tau} \mathbb{E}_{\pi}[c\tau + \Pi_\tau \wedge (1-\Pi_\tau)]$$



- The formulation with β is equivalent to the classical testing problem with c via a Lagrange formulation.
- In other words, the optimized Lagrange multiplier corresponds to $\frac{1}{c}$.

Remark 2. dependence on the signal-to-noise ratio

- In the one-match setting, if the observation process is given by $X_t = \theta\mu t + W_t$, then the signal-to-noise ratio is μ.
- By Brownian scaling, the process

$$\tilde{X}_t := \mu X_{t/\mu^2} = \mu\left(\theta\frac{t}{\mu^2} + W_{t/\mu^2}\right) = \theta t + \tilde{W}_t,$$

  where $\tilde{W}_t := \mu W_{t/\mu^2}$ is a standard Brownian motion.
- Then $T_{\mu,\beta}$ needed to achieve a certain precision $1-\beta$ when the signal-to-noise ratio is μ satisfies

$$T_{\mu,\beta} = \frac{1}{\mu^2}T_{1,\beta} = \frac{1}{\mu^2}T_\beta.$$

- Similarly, $\mathbb{E}[\tau_{\mu,\beta}] = \frac{1}{\mu^2}E[\tau_{1,\beta}] = \frac{1}{\mu^2}E[\tau_\beta]$.
- Both scale inversely with μ^2, but the average reduction ratio $R(\beta)$ is independent of μ. It suffices to consider $\mu = 1$.

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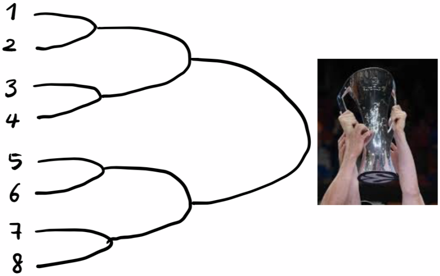
Problem formulation

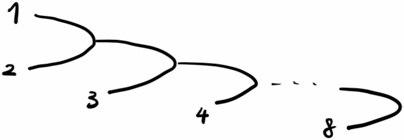
There are many different forms of tournaments. Of the knockout types:

- Football, table tennis,...



The champion!

- Children game, sorting:



The “king of the hill”

Problem formulation

- We consider a setting with 2^n players (of football type), each of whom has a distinct rank represented by an element of $\{1, 2, ..., 2^n\}$.
- We denote $f(j)$ the rank of Player j, and note that $f : \{1, 2, ..., 2^n\} \to \{1, 2, ..., 2^n\}$ is a bijection.
- A match between Player i and Player j is represented by a stochastic process

$$X_t^{ij} = \theta^{ij} t + W_t^{ij},$$

where W^{ij} is a standard Brownian motion and

$$\theta^{ij} = \begin{cases} 1 & \text{if } f(i) < f(j) \\ -1 & \text{if } f(i) > f(j). \end{cases}$$

- Thus X^{ij} is a Brownian motion with drift 1 if Player i has a better (smaller) rank than Player j, and it is a Brownian motion with drift -1 otherwise.

Problem formulation

- We assume a uniform prior distribution, i.e. every permutation of $\{1, 2, ..., 2^n\}$ is equally likely before any matches have been played.
- In each match, a winner qualifies to the next stage, whereas the loser is eliminated.
- Before the tournament starts, we specify the rules for how long the stochastic process X^{ij} should be observed. If $X^{ij} > 0$, player i is declared the winner.
- We assume each match is equally costly, and evaluate the cost criterion

$$\inf \mathbb{E}\left[\sum_{k=1}^{n} 2^{n-k}\tau_k\right]$$

where the inf is taken over tournaments such that

$$\mathbb{P}(\text{the strongest player also wins the tournament}) \geq \beta.$$

- In stage k, we use the same rule for all the games.

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Matches of fixed lengths

- We first consider a set-up where each match in stage k runs for a pre-determined time T_k.
- If $X^{ij}_{T_k} \geq 0$ in stage k, Player i is declared the winner.
- Assume that each match is equally costly, we consider

$$\inf_{\{T_k\}_{k=1}^{n}} \sum_{k=1}^{n} 2^{n-k} T_k$$

subject to the constraint

$$\mathbb{P}(\text{the strongest player wins the tournament}) \geq \beta, \quad \beta \in (2^{-n}, 1).$$

- Observe that the strongest player wins n matches of lengths $T_1, T_2, ..., T_n$ with probability

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Matches of fixed lengths

- We first consider a set-up where each match in stage k runs for a pre-determined time T_k.
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$$\inf_{\{T_k\}_{k=1}^n} \sum_{k=1}^{n} 2^{n-k} T_k \quad \text{subject to} \quad \prod_{k=1}^{n} \Phi(\sqrt{T_k}) \geq \beta. \tag{2}$$

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Theorem 2

Let $\beta \in (2^{-n}, 1)$ be given. Then there is a unique solution $(T_1, \ldots, T_n)$ with $T_k > 0$ for $k = 1, \ldots, n$ to the system

$$\begin{cases} \frac{\varphi(\sqrt{T_1})}{2^{n-1}\sqrt{T_1}\Phi(\sqrt{T_1})} = \frac{\varphi(\sqrt{T_2})}{2^{n-2}\sqrt{T_2}\Phi(\sqrt{T_2})} = \ldots = \frac{\varphi(\sqrt{T_n})}{\sqrt{T_n}\Phi(\sqrt{T_n})} \\ \prod_{k=1}^{n} \Phi(\sqrt{T_k}) = \beta, \end{cases}$$

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- For a fixed number of rounds n, the optimal time $T_k = T_k(\beta)$ is increasing in the precision level β, $k = 1, ..., n$.
- $T_1 < T_2 < \cdots < T_n$. i.e., later games should be played longer. It follows from the strict monotonicity of the function $F(u) := \frac{\varphi(u)}{u\Phi(u)}$.
- Equivalently,

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Fix $n \geq 2$ in a non-sequential tournament. For each $k \leq n-1$, the ratio $\frac{T_k}{T_{k+1}}$ is strictly increasing in β, with

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Sequential matches

- We specify an interval $(-b_k, b_k)$ for all matches in the kth stage, and run a match until the posterior probability process

$$\Pi_t^{ij} := \mathbb{P}(\theta_{ij} = 1 | \mathcal{F}_t^{ij})$$

leaves $(-b_k, b_k)$.

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Let $\beta \in (2^{-n}, 1)$ be given. Then there is a unique solution $(b_1, ..., b_n)$ with $b_k \in (0, 1/2)$ for $k = 1, ..., n$ to the system

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- Similarly, larger β gives longer average match length $\mathbb{E}[\tau_k]$ in stage k.
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Fix $n \geq 2$ in a sequential tournament. For each $k \leq n-1$, the ratio $\frac{\mu_k^*}{\mu_{k+1}^*}$ is strictly increasing in β, with

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Outline

What has been done so far?

- We have proved a within-design conclusion twice:

$$\beta_1^{fix} < \cdots < \beta_n^{fix}, \qquad \beta_1^{seq} < \cdots < \beta_n^{seq}.$$

- This says that for each optimized tournament, more information is allocated to later rounds **relative to itself**.

- To compare the designs, define two total costs

$$\begin{aligned} V_n^{fix}(\beta) &= \inf_{T_1,\ldots,T_n>0} \left\{ \sum_{k=1}^{n} 2^{n-k} T_k : \prod_{k=1}^{n} \Phi(\sqrt{T_k}) \geq \beta \right\}, \\ V_n^{seq}(\beta) &= \inf_{\mathbf{b}\in(0,\frac{1}{2})^n} \left\{ \sum_{k=1}^{n} 2^{n-k} \mu(b_k) : \prod_{k=1}^{n} (1-b_k) \geq \beta \right\}. \end{aligned} \tag{4}$$

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Cost comparison for a fixed budget β

Proposition 6 (tournament-level relative efficiency)

Fix n. For a tournament power $\beta \in (2^{-n}, 1)$, $\frac{1}{4} < \frac{V_n^{seq}(\beta)}{V_n^{fix}(\beta)} \leq R(\beta)$.

- In other words, sequential is always better, and the relative efficiency is bounded above by the one-match version for a fixed β.
- But we cannot reduce more than 75% by testing sequentially.
- In addition, one can show that V_n^{fix}, V_n^{seq} are strictly increasing and strictly convex in $\ln \beta$.

Proposition 7

Fix n. For a tournament power $\beta \in (2^{-n}, 1)$,

$$\lim_{\beta \downarrow 2^{-n}} \frac{\mu_k^*}{T_k^*} = \frac{2}{\pi}, \quad \lim_{\beta \uparrow 1} \frac{\mu_k^*}{T_k^*} = \frac{1}{4}.$$

- Follows from asymptotic analysis in the two schemes.
- However, for a fixed n and k, the monotonicity of the cross ratio $\frac{\mu_k^*}{T_k^*}$ in β is more difficult. Related to how the "accuracy budget" is allocated.

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Allocation comparison for a fixed budget β

Because the two optimized accuracy vectors have the same product,

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the sequential accuracies cannot be larger in **every round** unless the two vectors are identical.

Question

Does the sequential scheme shift accuracy toward later rounds relative to fixed-length scheme?

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There exists a cutoff $j \in \{1, \dots, n-1\}$ such that

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Thus $\beta_k^{seq} - \beta_k^{fix}$ changes sign exactly once, from negative to positive.

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Remark 3. stage-dependent costs and relative strength

Recall that we had many symmetry assumptions:

- In the matches of different players, we always observe the same strength difference, 1 or -1.
- Each match is equally costly.
- We always play the same rule in the same stage.
- We start from a uniform prior.
- The tree itself is symmetric.

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Remark 3. stage-dependent costs and relative strength

- Later matches might be more expensive than earlier ones:

$$c_1 \leq c_2 \leq \cdots \leq c_n.$$

- Later round relative strengths might be tighter than earlier ones:

$$X_t^{ij} = \mu_k \theta^{ij} t + W_t^{ij},$$

  where $\mu_1 \geq \cdots \geq \mu_k$. (Later games might give smaller signals)

- Define the stage price of information as $L_k := 2^{n-k} \frac{c_k}{\mu_k^2}$.
- If the inequality $L_{k+1} \leq L_k$ holds for all $1 \leq k \leq n$, all results still hold.

Remark 3. stage-dependent costs and relative strength

- Later matches might be more expensive than earlier ones:

$$c_1 \le c_2 \le \cdots \le c_n.$$

- Later round relative strengths might be tighter than earlier ones:

$$X_t^{ij} = \mu_k \theta^{ij} t + W_t^{ij},$$

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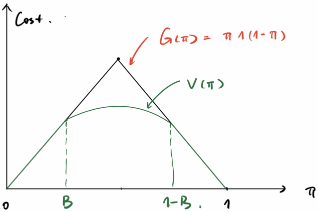
1. What if we allow matches in the **same stage** k to use **asymmetric strategies**?

Proposition 9

Assume that the stopping strategy $(b_1, 1-b_1), (b_2, 1-b_2)$ *are applied in one game, the total cost*

$$\mathbb{E}[\tau(b_1) + \tau(b_2)] \geq 2\mathbb{E}[\tau(\frac{b_1+b_2}{2})].$$

2. What if we have the **“king of the hill”** tournament scheme?
The symmetry is immediately broken, and in terms of Shiryaev,



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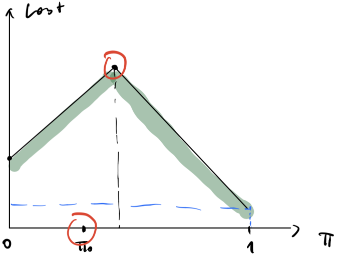

Remark 4. how do we deal with asymmetry

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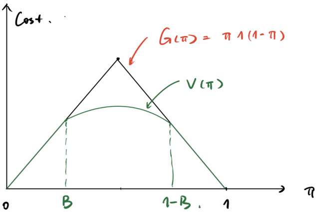

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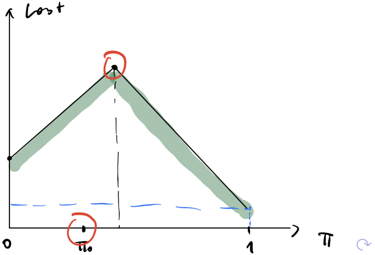

3. What if we have arbitrary prior on the $n!$ configurations?
   - e.g. $n = 2$. Assume that Player 2 and 4 got eliminated in the semi-final at observing $X^{12}_{\tau_{12}} = x_{12}, X^{34}_{\tau_{34}} = x_{34}$.
   - In general, the "regret" in the final

$$\mathbb{P}(\text{Player 2 or 4 are not the best} | x_{12}, x_{34}, X^{13}_t)$$

     is not the same as

$$\mathbb{P}(\text{Player 2 or 4 are not the best} | x_{12}, x_{34}).$$

   - In other words, the regret is not a constant, and things become very asymmetric.

Next step: solve examples of 2 and 3.

Thank you!