Thompson Sampling Algorithm for Stochastic Differential Games

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Joint work with Asaf Cohen and Ruolan He

Byrne Conference on Stochastic Analysis in Finance and Insurance

June 3, 2026

Motivation

In some games, many agents respond to a common signal that no one knows exactly.

- **Regulatory signal**
- **Market signal**

We consider a $N-$player competitive stochastic game where player i controls a private diffusion $X^i \in \mathbb{R}^d$ with the common unknown drift A:

$$dX_t^i = (AX_t^i - \alpha_t^i)\,dt + \sigma^i\,dW_t^i, \quad i = 1, \ldots, N.$$

- a common unknown drift $A \in \mathbb{R}^{d\times d}$ entering every agent's dynamics,
- each agent observes only its own trajectory, all W's are independent, and indpendent of A. They must act while learning;
- agents compete through costs, not by touching each other's state;
- an ergodic quadratic cost:

$$J^i(\boldsymbol{X}_0, \alpha) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^i\Big[\int_0^T \Big((\boldsymbol{X}_t - \overline{\boldsymbol{x}}_i)^\top \mathcal{Q}^i(\boldsymbol{X}_t - \overline{\boldsymbol{x}}_i) + \frac{1}{2}(\alpha_t^i)^\top R^i \alpha_t^i\Big)\,dt\Big]$$

Questions

- Can we find learning-based strategies that give a Nash Equilibrium?
- What is the price of not knowing A?

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Our contributions

Background:

- Ergodic LQ N-player games with *known* A: explicit affine Nash feedbacks (Bardi–Priuli '14).
- Filtering of an unknown drift.
- Thompson sampling (TS) for *single-agent* LQ control: $O(\sqrt{T \log T})$ regret (Ouyang et al. '17/'20, Gagrani et al. '21).

What we do:

- Carry TS from single-agent control to an N-player game with partial information.
- Regret of the same order $O(\sqrt{T \log T})$ independent of N, under weaker assumptions.
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The full-information game (Bardi–Priuli, 2014)

- Assume A is known to all players.
- Set the Hamiltonian for Player i:

$$H^i(x,p) := \max_{a\in\mathbb{R}^d}\Big\{-\frac{1}{2}a^\top R^i a - p^\top(Ax-a)\Big\} = \frac{1}{2}p^\top (R^i)^{-1}p - p^\top Ax,$$

- Using ergodicity, integrate the others out against their stationary laws $\boldsymbol{m}^{-i}$:

$$\widetilde{f}^i(x;\boldsymbol{m}^{-i}) = \int \widetilde{F}^i(\xi^1,\dots,x,\dots,\xi^N)\prod_{j\neq i} dm^j(\xi^j).$$

- Denote $\varsigma^i = \frac{1}{2}(\sigma^i)(\sigma^i)^T$. Then the stationary HJB–KFP system associated with the game is:

$$\boxed{\begin{aligned}
&-\operatorname{tr}(\varsigma^i D^2 v^i(x)) + H^i(x,\nabla v^i(x)) + \lambda^i = \tilde{f}^i(x;\boldsymbol{m}^{-i}),\\
&-\operatorname{tr}(\varsigma^i D^2 m^i(x)) - \operatorname{div}\left(m^i(x)\frac{\partial H^i}{\partial p}(x,\nabla v^i(x))\right) = 0,\\
&\int_{\mathbb{R}^d} m^i(x)\,dx = 1, \quad m^i(x) > 0.
\end{aligned}}$$

The full information game (Bardi–Priuli, 2014)

- Under some assumptions, the HJB–KFP system has a unique solution, explicit in A:

$$v^i(x) = \tfrac{1}{2} x^\top \Lambda^i x + (\rho^i)^\top x, \qquad m^i = \mathcal{N}\big(\eta^i, (\Upsilon^i)^{-1}\big), \qquad \lambda^i \in [0, \infty),$$

- where $\Upsilon^i(A)$ solves the Riccati equation $\frac{1}{2}\,\Upsilon\,\varsigma^i R^i \varsigma^i\,\Upsilon = \frac{1}{2} A^\top R^i A + Q^i_{ii}$.
- Affine Nash feedback

$$\bar{a}^i(x; A) = \big(\varsigma^i \Upsilon^i(A) + A\big)x - \varsigma^i \Upsilon^i(A)\,\eta^i(A).$$

- The resulting state process is Ornstein–Uhlenbeck:

$$dX^i_t = \big(-\varsigma^i \Upsilon^i X^i_t + \varsigma^i \Upsilon^i \eta^i\big)\,dt + \sigma^i\,dW^i_t..$$

Notations

- No hat = truth, full info. e.g., A, X^i_t, α^i_t
- Hat = sampled. e.g., $\widehat{A}^i_k, \widehat{X}^i_t, \widehat{\alpha}^i_t$.

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Partial information game: A unknown

- Now consider A unknown. Not solvable.
- **Goal:** Find an equilibrium strategy profile $\hat{\alpha}$ with a efficient learning of the parameter such that as $T \to \infty$, the players will behave like in the ergodic game with full information. We use the Thompson Sampling (TS) algorithm.
- Idea of the algorithm: sample a reasonable $\widehat{A}$, then play $\bar{a}^i(\cdot; \widehat{A})$ as if $\widehat{A}$ were true.

Prior Distribution of the Unknown Drift A

For any $i \in [N]$,

$$A \sim \mathcal{N}(\mu_0^i, \Sigma_0^i) \quad |_\Theta$$

- Θ is a compact set.
- Assume boundedness, stability. Relax on independence.

Each Player i Maintains a Posterior $\mathcal{N}(\mu_t^i, \Sigma_t^i)$ Over A:

- Updates posterior using its own observed trajectory $(\hat{X}_t^i)_{t \geq 0}$
- Filtering is done in real time, independently by each player

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Thompson Sampling Algorithm

Initialization: Each player initializes prior (μ_0^i, Σ_0^i) and samples $\hat{A}_0^i \sim \mathcal{N}(\mu_0^i, \Sigma_0^i)\,|_{\Theta^{(v)}}$.

1. During episode k, observe the state dynamics $\hat{X}_t^i$, by using the strategy $\hat{\alpha}_t^i = \bar{a}^i(\hat{X}_t^i; \hat{A}_k^i)$ via the latest sampled $\hat{A}_k^i$.
2. Update posterior $\mathcal{N}(\mu_t^i, \Sigma_t^i)$.
3. **(Stopping criteria)**: If either
   - Interval length stopping criterion: $t - t_k^i \geq T_{k-1}^i + 1$
   - Determinant stopping criterion: $\det(\Sigma_t^i) < 0.5 \det(\Sigma_{t_k^i}^i)$

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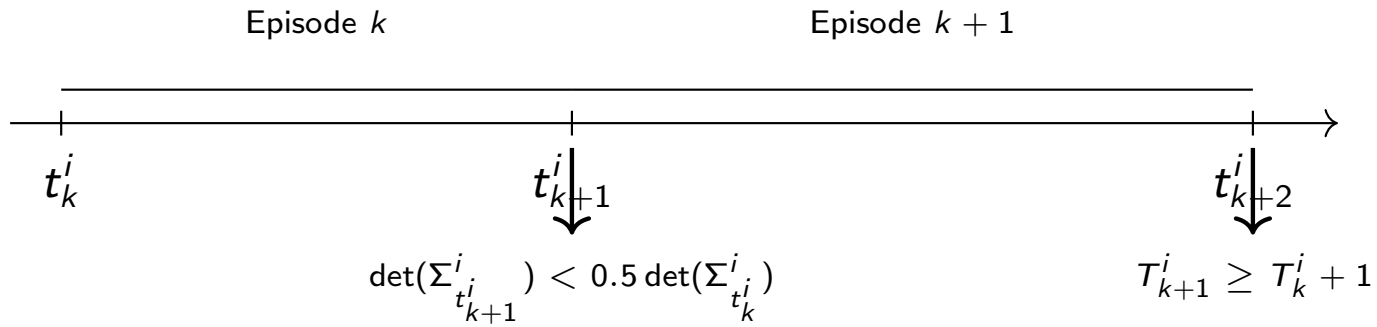

Two intermediate estimates: learning and coupling

1. Parameter learning estimate

$$\int_0^T \mathbb{E}^i \|A - \widehat{A}^i_{k(t)}\|^2 \, dt \leq O(\log T).$$

- Heuristically, the posterior covariance decreases $\sim 1/t$
- This estimate controls how wrong the sampled model is over time.

2. State coupling estimate

$$\int_0^T \mathbb{E}^i \|\widehat{X}^i_t - X^i_t\|^2 \, dt \leq O(\sqrt{T \log T}).$$

- Since $\widehat{A}$ is close to A, and the feedback coefficients depend regularly on A, the sampled feedback is close to the true Nash feedback.
- Therefore the learned trajectory is close to the full-information trajectory:

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Main Theorem: Regret Bounds

Definition

The **Regret** of Player i up to time T (against the full information cost) is:

$$R^i(\hat{\alpha}^i, T) := \mathbb{E}\left[\int_0^T f^i(\hat{X}_t^i, \hat{\alpha}_t^i; \boldsymbol{m}^{-i})dt - T\lambda^i(A)\right]$$

Theorem (Regret bound (Cohen, He, W.))

Player i's regret $R^i(\widehat{\alpha}^i, T) \leq O\left(\sqrt{T \log T}\right)$.

- The regret bound is uniform with respect to the number of players N.
- We decompose

$$R^i = \underbrace{\mathbb{E}^i \int_0^T (\lambda^i(\widehat{A}_{k(t)}) - \lambda^i(A))\, dt}_{R_0 \text{ sampling error}} + \underbrace{\mathbb{E}^i\left[v^i(X_0) - v^i(\widehat{X}_T)\right]}_{R_1 \text{ cumulative strategy effect}} + \underbrace{\mathbb{E}^i \int_0^T (\nabla v^i(\widehat{X}_t))^\top (A - \widehat{A}_{k(t)})\widehat{X}_t\, dt}_{R_2 \text{ model mismatch}}.$$

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Theorem (Nash Equilibrium (Cohen, He, W.))

If every player runs TS, then $(\widehat{\alpha}^1, \ldots, \widehat{\alpha}^N)$ is a Nash equilibrium, and each $\widehat{X}^i$ is ergodic with the same limiting distributions m^i as in the full information case.

Three steps Fix player i:

1. **Admissibility & ergodicity:** $\widehat{X}^i$ inherits ergodicity and the law m^i of the full information case.
2. **Cost under TS $= \lambda^i$:** splitting state and control terms, the gap to the benchmark cost vanishes in time average; the remainder equals λ^i through the HJB identity.
3. **No unilateral deviation:** since dynamics are not coupled.

Remark. The control itself converges sublinerally.

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Numerical experiments

Experiment Setup:

- **State dimension:** $d = 2$; **Number of players:** $N = 10$
- **True drift matrix:** $A = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}$
- **Noise:** $\sigma^i = 0.5 \cdot I_d + 0.05 \cdot Z^i, \quad Z^i \sim \mathcal{N}(0, I_d)$, independently sampled.
- **Prior:** $A \sim \mathcal{N}(0, 0.5I)$
- **Initial State:** $X_0^i = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$.

Questions:

1. Does learning/state/control error and the regret scale as they should?
2. Does TS beat certainty-equivalent (posterior-mean) and blind baselines?
3. Is it robust to prior mis-specification?

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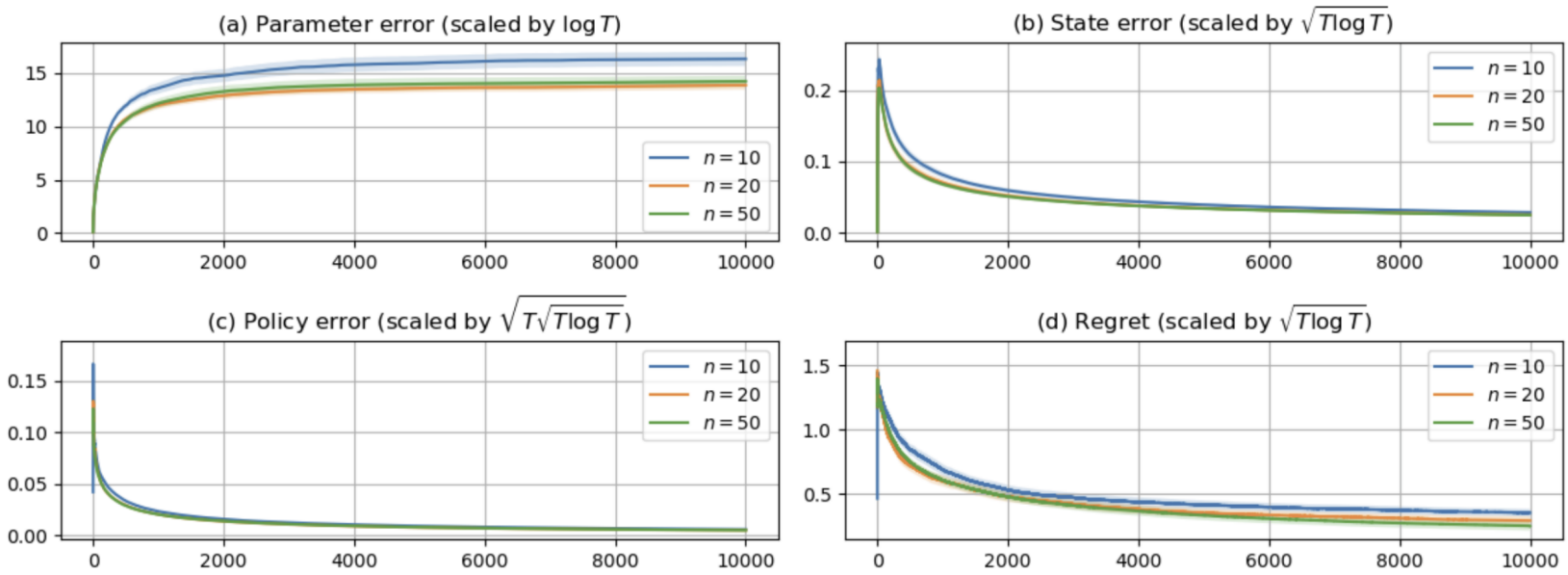
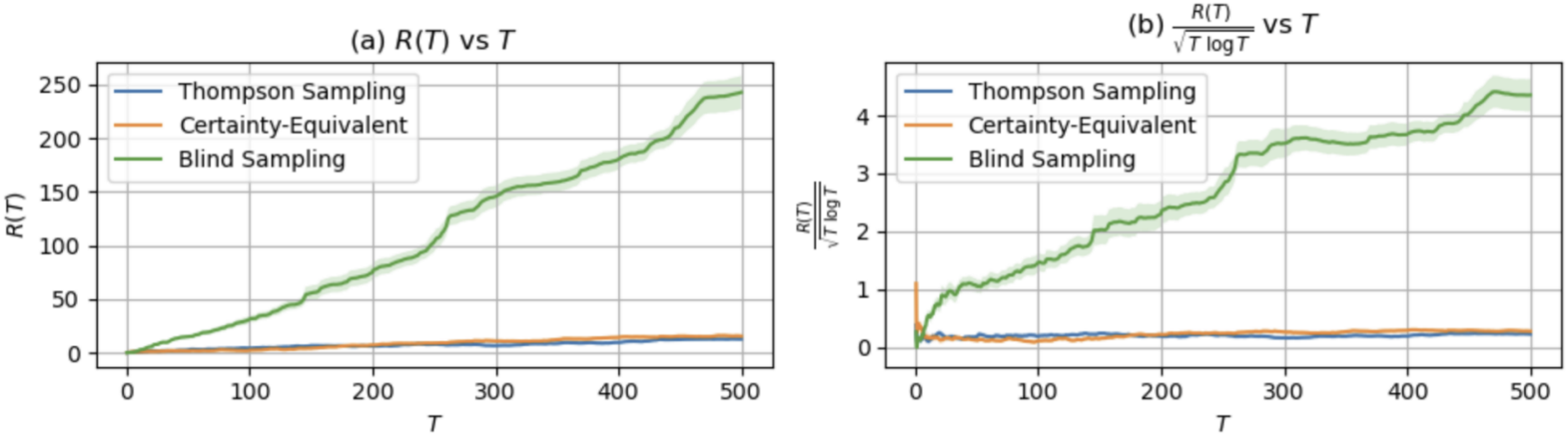
Numerics 1: learning, coupling, and regret



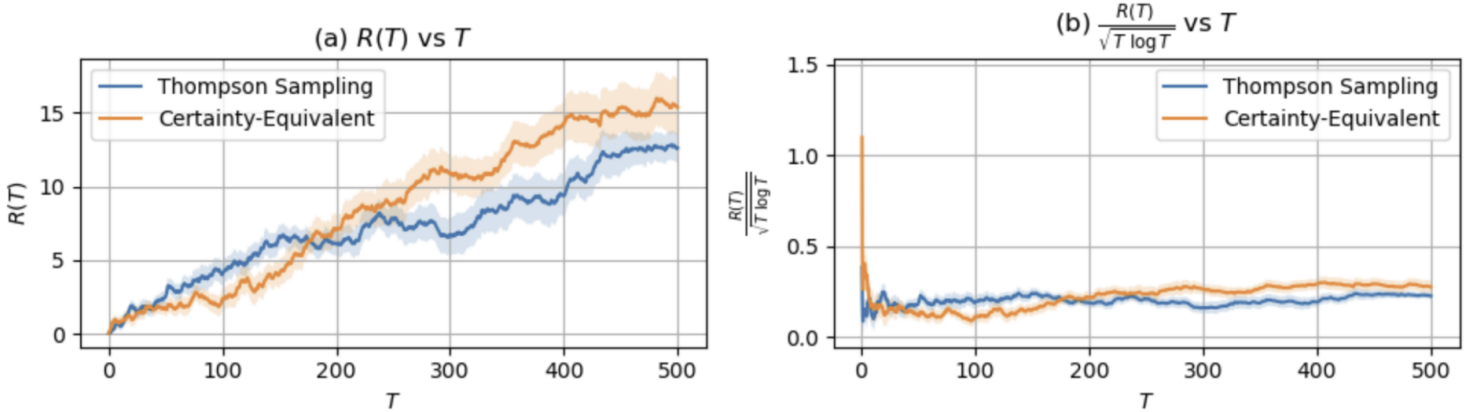
- **Learning:** parameter error scaled by $\log T$ stabilizes, consistent with $\int_0^T \mathbb{E}\|A - \widehat{A}_{k(t)}\|^2 dt = O(\log T)$.
- **Coupling:** state and policy errors decay after normalization, so TS tracks the full-information benchmark in time average.
- **Regret:** regret scaled by $\sqrt{T \log T}$ stays bounded/decreases across sample sizes $n = 10, 20, 50$.

Numerics 2: Thompson sampling vs certainty equivalent

With blind sampling



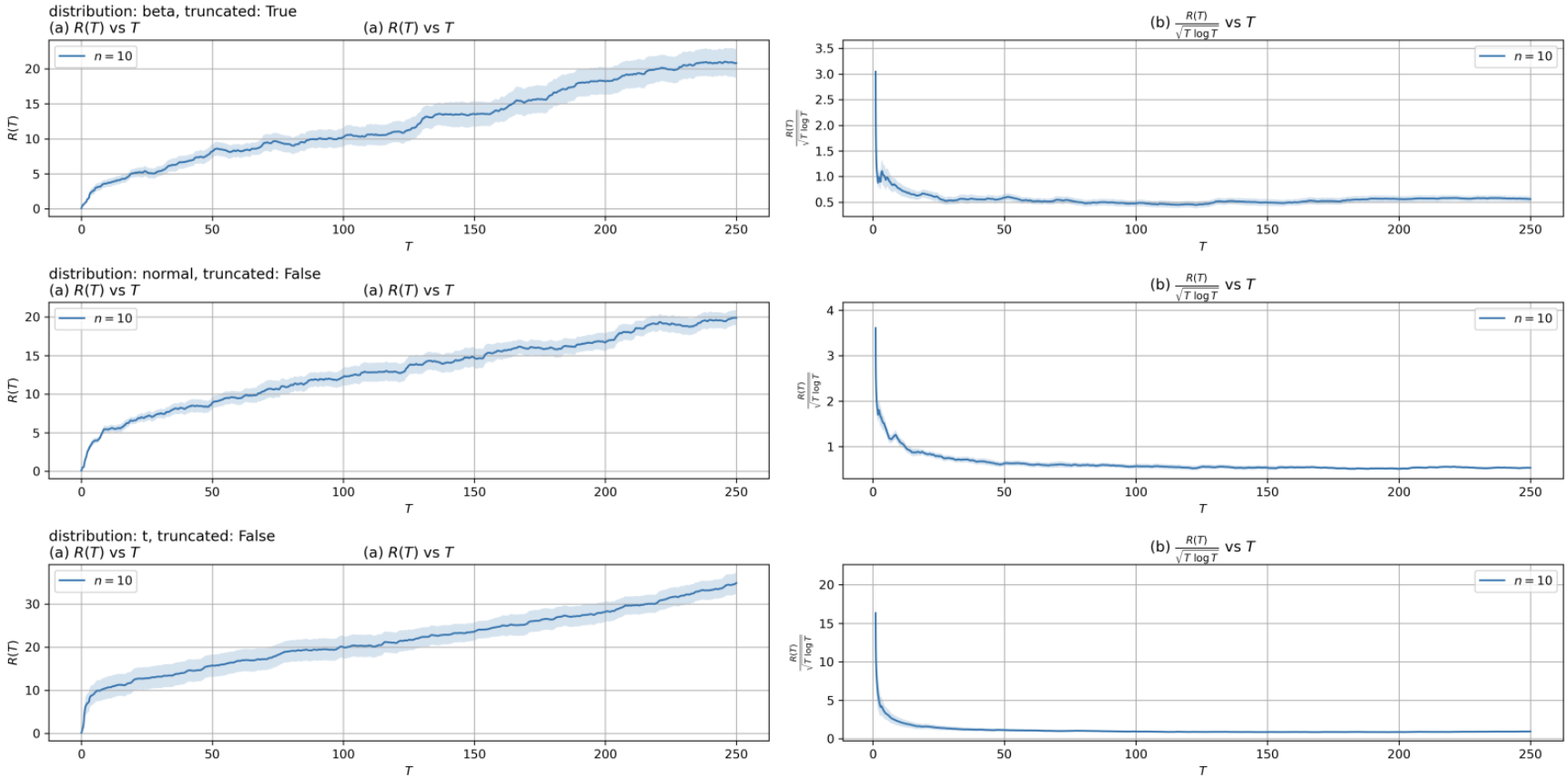
Thompson sampling vs certainty-equivalent (plug in μ_t^i)



- Blind sampling is much worse.
- CE strategy has slightly smaller regret earlier due to greedy exploitation, but fails exploration and accumulates larger regret.

Numerics 3: Robustness to prior mis-specification

Regret under different priors



- Same asymptotic scaling for Gaussian, t, exponential, beta.
- Same asymptotic scaling for both truncated and not truncated.

Future directions

- **Coupling through dynamics**, one player's learning/control affects another player's observations
- **Closed-loop games:** players observe the *full* state vector and react to each other

Thank you!