Play longer when It matters: optimal match length in knock-out tournaments

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Outline

Motivation: knock-out tournament design

The goal of a tournament is to single out the best player. Sometimes we think later stage games are more important:

- **Table tennis/badminton.** Early rounds are often best of 3 sets, while semifinals/finals are 5 or 7 sets.
- **Boxing.** Title fights are scheduled for 12 rounds, while non titled 4-10 rounds

Does this make sense intuitively? The final determines the champion, while earlier rounds are numerous and each is less “informative” for the final winner.

Does it make sense statistically?

Motivation: knock-out tournament design

In this talk we study how to optimally allocate match lengths in a 2^n-player knock-out tournament.

Two design questions:

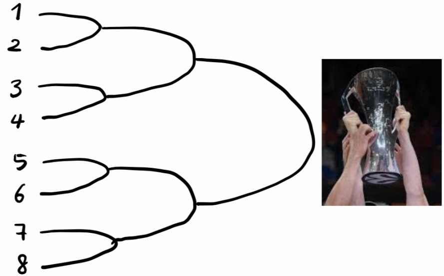
1. Should later stage games be longer?
2. **Round allocation:** If yes, what is the *optimal round-dependent schedule*? Are the optimal designs for sequential and fixed-length tournaments different?

Outline

Problem formulation

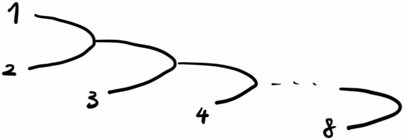
There are many different forms of tournaments. Of the knockout types:

- Football, table tennis,...



The champion!

- Children game, sorting:



The “king of the hill”

Problem formulation

- We consider a setting with 2^n players (of football type), each of whom has a distinct rank
- A match between Player i and Player j is represented by a stochastic process

$$X_t^{ij} = \theta^{ij} t + W_t^{ij},$$

where W^{ij} is a standard Brownian motion and

$$\theta^{ij} = \begin{cases} 1 & \text{if player i is better} \\ -1 & \text{if player j is better.} \end{cases}$$

- We assume each match is equally costly, and evaluate the cost criterion

$$\inf \mathbb{E}\left[\sum_{k=1}^{n} 2^{n-k}\tau_k\right]$$

where the inf is taken over tournaments such that

$$\mathbb{P}(\text{the strongest player also wins the tournament}) \geq \beta.$$

- In stage k, we use the same rule for all the games.

Matches of fixed lengths

- We first consider a set-up where each match in stage k runs for a pre-determined time T_k.
- If $X^{ij}_{T_k} \geq 0$ in stage k, Player i is declared the winner. Probability i is actually the strong one: $\Phi(T_k)$.
- Assume that each match is equally costly, we consider

$$\inf_{\{T_k\}_{k=1}^n} \sum_{k=1}^n 2^{n-k} T_k \quad s.t. \quad \mathbb{P}(\text{the best player wins the tournament}) \geq \beta,$$

  for $\beta \in (2^{-n}, 1)$.
- Observe that the strongest player wins n matches of lengths T_1, $T_2, \ldots, T_n$ with probability

$$\prod_{k=1}^n \Phi(\sqrt{T_k}).$$

Matches of fixed lengths

- The optimization problem can be written as

$$\inf_{\{T_k\}_{k=1}^n} \sum_{k=1}^{n} 2^{n-k} T_k \quad \text{subject to} \quad \prod_{k=1}^{n} \Phi(\sqrt{T_k}) \geq \beta. \tag{1}$$

- Define the Lagrangian
$\mathcal{L}(\{T_k\}_{k=1}^n, \lambda) := \sum_{k=1}^n 2^{n-k} T_k - \lambda \left(\prod_{k=1}^n \Phi(\sqrt{T_k}) - \beta\right)$, $\lambda \geq 0$.

Theorem 1

Let $\beta \in (2^{-n}, 1)$ be given. Then there is a unique solution $(T_1, ..., T_n)$ with $T_k > 0$ for $k = 1, ..., n$ to the system

$$\begin{cases} \frac{\varphi(\sqrt{T_1})}{2^{n-1}\sqrt{T_1}\Phi(\sqrt{T_1})} = \frac{\varphi(\sqrt{T_2})}{2^{n-2}\sqrt{T_2}\Phi(\sqrt{T_2})} = ... = \frac{\varphi(\sqrt{T_n})}{\sqrt{T_n}\Phi(\sqrt{T_n})} \\ \prod_{k=1}^n \Phi(\sqrt{T_k}) = \beta, \end{cases}$$

and $(T_1, ..., T_n)$ solves the (1), where $T_1 < T_2 < \cdots < T_n$. Equivalently, $\beta_1^{fix} < \beta_2^{fix} < \cdots < \beta_n^{fix}$, with $\prod_{k=1}^n \beta_k^{fix} = \beta$.

Sequential matches

- We specify an interval $(-b_k, b_k)$ for all matches in the kth stage, and run a match until the posterior probability process

$$\Pi_t^{ij} := \mathbb{P}(\theta_{ij} = 1 | \mathcal{F}_t^{ij})$$

  leaves $(-b_k, b_k)$.
- The match is played until $\tau_k^{ij} := \tau^{ij}(b_k) := \inf\{t \geq 0 : \Pi_t^{ij} \notin (b_k, 1 - b_k)\}$, and Player i is declared winner if $\Pi_{\tau_k}^{ij} = 1 - b_k$.
- For each rule $b_k \in (0, \frac{1}{2})$, we have that

$$\mathbb{P}(\text{the better player wins the match}) = 1 - b_k.$$

- Denote the expected value of observation time τ_k, $\mu(b_k) := \mathbb{E}[\tau_k]$. The constrained sequential tournament design problem becomes

$$\inf_{\mathbf{b} \in (0, \frac{1}{2})^n} \sum_{k=1}^{n} 2^{n-k} \mu(b_k) \quad \text{subject to} \quad \prod_{k=1}^{n} (1 - b_k) \geq \beta. \tag{2}$$

Sequential matches

- Define for $\lambda \geq 0$ the Lagrangian $\mathcal{L}(\mathbf{b}, \lambda) := \sum_{k=1}^{n} 2^{n-k}\mu(b_k) - \lambda \left(\prod_{k=1}^{n}(1-b_k) - \beta\right)$.
- The first order conditions yields

Theorem 2

Let $\beta \in (2^{-n}, 1)$ be given. Then there is a unique solution $(b_1, ..., b_n)$ with $b_k \in (0, 1/2)$ for $k = 1, ..., n$ to the system

$$\begin{cases} 2^{n-1}(1-b_1)\mu'(b_1) = 2^{n-2}(1-b_2)\mu'(b_2) = ... = (1-b_n)\mu'(b_n), \\ \prod_{k=1}^{n}(1-b_k) = \beta. \end{cases} \tag{3}$$

Moreover, $(b_1, ..., b_n)$ solves the optimization problem (2), and $b_1 \geq b_2 \geq \cdots \geq b_n$. Equivalently, $\beta_1^{seq} < \beta_2^{seq} < \cdots < \beta_n^{seq}$, $\prod_{k=1}^{n} \beta_k^{seq} = \beta$.

Outline

Allocation comparison for a fixed budget β

Proposition 3

There exists a cutoff $j \in \{1, \dots, n-1\}$ such that

$$\beta_k^{seq} < \beta_k^{fix} \quad (k < j),$$

$$\beta_j^{seq} \le \beta_j^{fix},$$

and

$$\beta_k^{seq} > \beta_k^{fix} \quad (k > j).$$

Thus $\beta_k^{seq} - \beta_k^{fix}$ changes sign exactly once, from negative to positive.

- i.e., the sequential scheme tolerates more earlier round randomness and compensates with stricter later round decisions.
- The same comparison can be stated cumulatively: $\prod_{k=1}^{r} \beta_k^{seq} < \prod_{k=1}^{r} \beta_k^{fix}$, for $r = 1, \dots, n-1$, while equality holds at n.

Outline

What assumptions are essential?

Recall that we had many symmetry assumptions:

- In the matches of different players, we always observe the same strength difference, 1 or -1.
- Each match is equally costly.
- We always play the same rule in the same stage.
- We start from a uniform prior.
- The tree itself is symmetric.

What results still hold if we relax these assumptions?

What assumptions are essential?

Blue: can relax / is true, Red: yields a different problem.

- In the matches of different players, we always observe the same strength difference, 1 or -1.
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- The tree itself is symmetric.

Thank you!