How to walk home tipsy: from random walks to PDEs

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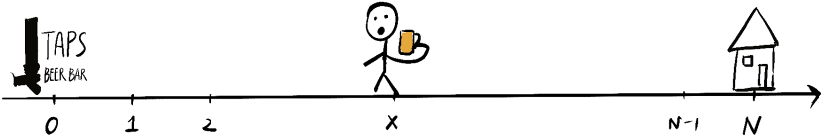
Outline

Problem Setting



Questions to ask:

1. What is the probability that you find yourself home before ending up in Taps again?
2. What is the expected time of doing so?
3. What is the probability distribution of the walker?

A Simple Random Walker on $\mathbb{Z}^d$

Consider a random walker on $\mathbb{Z}^d$ starting from position x_0: at each integer time, the walker takes one step to one of his $2d$ neighbours with equal probability $\frac{1}{2d}$, independent of the past.

Let S_n denote his position at time n:

$$S_n = x_0 + X_1 + \cdots + X_n.$$

Probability distribution of the walker in 1d

W.l.o.g, assume $x_0 = 0$. Denote the probability that the walker is at position x at time n by $p_n(x)$.

Consider now in a higher dimension: why birds do not drink

"A drunk man will find his way home, a drunk bird may get lost forever."

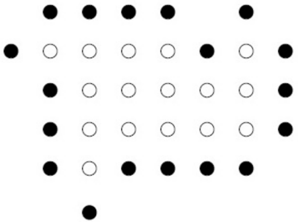
- Consider $p_{2n}(0)$.
- Central Limit Theorem says that $\frac{S_n}{\sqrt{n}}$ converges to a multivariate normal vector.
- In 1d, there are $O(\sqrt{n})$ comparable nodes.

SRW on a bounded subset of $\mathbb{Z}^d$

- Consider a connected subset $A \subset \mathbb{Z}^d$. Its boundary ∂A is the set of points in $\mathbb{Z}^d$ that are adjacent to a point in A:

$$\partial A := \{x \in \mathbb{Z}^d \backslash A : |y - x| = 1, \text{ for some } y \in A\}.$$

- Let $\bar{A} = A \cup \partial A$ be the discrete closure of A.
- Let $\tau_A := \min\{n \geq 0 : S_n \notin A\}$ be the first time that the SRW hits the boundary.



Take it to 1d and answer Question 1

Consider the tipsy walker. Let $\bar{A} = \{0, 1, \ldots, N\}$, and let S_n be a 1d SRW starting from $x \in A$. What is the probability that the walker reaches N before 0?

Guess:

First sol:

Take it to 1d and answer Question 1

Second sol: Alternatively, let us write $f(x) = \mathbb{P}_x(S_{\tau_A = N})$.

Some Terms

Discrete Laplacian

The Laplacian on the graph $\mathbb{Z}^d$ is the operator Δ defined by

$$\Delta f(x) = \frac{1}{2d} \sum_{y \sim x} f(y) - f(x).$$

Harmonic Function

A function $f : \bar{A} \to \mathbb{R}$ is said to be harmonic if $\Delta f = 0$ for all $x \in A$.

Weak Maximum Principle

A harmonic function $f : \bar{A} \to \mathbb{R}$ achieves its extrema on ∂A.

Two comments:

Question 1 is equivalent to

Discrete Dirichlet Problem

Given a graph A and boundary ∂A. Find the unique harmonic function $f : \bar{A} \to \mathbb{R}$ such that $f|_{\partial A} = F$.

An Example

The stopping time τ_A is finite with probability 1.

Another Definition

Discrete Harmonic Measure

Let $B \subset \partial A$. What is the probability $H(x, B)$ that a SRW start from $x \in A$ hits B?

- On the boundary: $H(x, B) = 1$ for $x \in B$ and $H(x, B) = 0$ for $x \in \partial A \backslash B$
- On the interior: $H(x, B) = \frac{1}{2d} \sum_{y \sim x} H(y, B)$.

Solution to the Discrete Dirichlet Problem

Let $H_A(x, y) = \mathbb{P}(S_{\tau_A} = y), y \in \partial A$. ("Poisson Kernel")

Theorem

The unique solution to

$$\begin{cases} f(x) = F, \text{ on } \partial A, \\ \Delta f = 0, \text{ in } A \end{cases}$$

is

$$f(x) = \mathbb{E}_x[F(S_{\tau_A})] = \sum_{y \in \partial A} H_A(x, y) F(y).$$

Answer to Question 2

In 1d, What is the expected time the walker takes to reach site 0 or site N?

First sol: In d-dim, consider process $M_n := |S_{n \wedge \tau_A}|^2 - (n \wedge \tau_A)$.

Answer to Question 2

Second sol: Let $f(x) = \mathbb{E}_x[\tau_A]$. Then:

$$\begin{cases} f(x) = 0, x \in \partial A, \\ \Delta f(x) = -1, x \in A. \end{cases}$$

Note that

- $\Delta(-x^2) = -1$,
- g(x) = x is harmonic.

Another Definition

Green's function of SRW

For any $y \in A$, let V_y denotes the number of visits to y before leaving A.

$$V_y = \sum_{n=0}^{\infty} 1_{S_n = y, \tau_A > n},$$

$$\mathbb{E}_x[V_y] = \sum_{n=0}^{\infty} \mathbb{P}_x(S_n = y, \tau_A > n) =: G_A(x, y).$$

Another Definition

Fix $y \in A$, the Green's function of SRW satisfies

$$\Delta G_A(x,y) = \begin{cases} -1, x = y, \\ 0, x \neq y, . \end{cases}$$

In $d \geq 3$, we can define the whole domain Green's function, and it is bounded:

$$G(x,y) = \lim_{A \to \mathbb{Z}^d} G_A(x,y).$$

Question 2 is equivalent to: Discrete Poisson Equation

Theorem

Let $\rho : A \to \mathbb{R}$ *be a given function, then the unique* $f : \bar{A} \to \mathbb{R}$ *which solves*

$$\begin{cases} f(x) = 0, x \in \partial A, \\ \Delta f(x) = -\rho, x \in A. \end{cases}$$

can be written as

$$f(x) = \mathbb{E}_x[\sum_{n=0}^{\tau_A - 1} \rho(S_n)] = \sum_{y \in A} G_A(x,y)\rho(y).$$

Discrete Poisson Equation

Theorem

Let $\rho : A \to \mathbb{R}$ and $F : \partial A \to \mathbb{R}$ be given functions, then the unique $f : \bar{A} \to \mathbb{R}$ which solves

$$\begin{cases} f(x) = F, x \in \partial A, \\ \Delta f(x) = -\rho, x \in A. \end{cases}$$

can be written as

$$f(x) = \mathbb{E}_x[F(S_{\tau_A})] + \mathbb{E}_x[\sum_{n=0}^{\tau_A - 1} \rho(S_n)] = \sum_{z \in \partial A} H_A(x,z)F(z) + \sum_{y \in A} G_A(x,y)\rho(y).$$

Back to Question 3

The probability of being at position x at time $n+1$:

$$p_{n+1}(x) = \frac{1}{2}p_n(x-1) + \frac{1}{2}p_n(x+1).$$

"Discrete Heat equation", analogue of

$$u_t = \frac{1}{2}u_{xx}.$$

Discrete Heat Equation

Consider the "time-limited" harmonic measure

$$H_{A,t}(x,y) = \mathbb{P}_x(S_{\tau_A \wedge t} = y).$$

Theorem

The unique solution $f : \bar{A} \to \mathbb{R}$ which solves the following "discrete heat equation"

$$\begin{cases} \Delta f(x,t) = f(x,t+1) - f(x,t), & \textit{for } (x,t) \in A \times \mathbb{N} \\ f(x,t) = F(x), & \textit{for} (x,t) \in \partial A \times \mathbb{N} \cup A \times \{0\}, \end{cases}$$

is

$$f(x) = \sum_{y \in \bar{A}} H_{A,t}(x,y) F(y).$$

A Proof of the Central Limit Theorem (Petrovsky and Kolmogorov)

Idea of the proof

- Let X_i be i.i.d. with mean 0 and variance 1, and let $U_n(x)$ be the distribution function of $\sum_{j=1}^{n} \frac{X_j}{\sqrt{n}}$.
- Want: $\lim_{n\to\infty} U_n = \phi$.
- Observe that $\phi(\frac{x}{\sqrt{t}})$ solves the heat equation $u_t = \frac{1}{2}u_{xx}$ on the half plane $t > 0$,
- Let $v(x,t) = \phi(\frac{x}{\sqrt{t}}) + \epsilon t$, then v solves $v_t = \frac{1}{2}u_{xx} + \epsilon$. "Upper Function".
- Every step we substitute U_n with $\phi(\sqrt{n}x)$. The error in each step is small enough that the overall error is negligible.
- For sufficiently large n,

$$U_n(x) < \phi(x) + 2\epsilon.$$

Thank you!