Uppsala Universitet
Matematiska Institutionen
I. Kaj, R. Müller, Y. Wang

Repetition and important formulas from modul 10

1 Random processes and their properties

Stochastic processes model dynamically changing random quantities.

Stochastic process A stochastic process (or random process) $\{X(t), t \in I\}$ is a collection of random variables indexed by a set I. We distinguish stochastic processes with

- discrete time (in this case the stochastic process is often called "time series"): $X_0, X_1, X_2, \ldots$ or $\ldots, X_{-1}, X_0, X_1, \ldots$, i.e. I is a sequence of integers
- continuous time: $X_t, t \geq 0$ or $X_t, -\infty < t < \infty$, i.e. I is an interval of the line

The set E of possible values for $X(t)$ is called state space. A stochastic process has

- discrete states, if E is a discrete set such as $\{0, 1, \ldots, r\}$ or $\{0, 1, \ldots, \}$; and
- continuous states, if E is an interval of the real line such as $E = [0, \infty)$.

Examples Several standard examples of discrete time random processes are discussed in the notes:

- Random walk (example 2.2.)
- Galton's Bean Machine (example 2.3.)
- Partial sum process (example 2.4.)

Properties With a stochastic process $\{X(t)\}$ we can associate

- the mean value function $m(t) = \mathrm{E}[X(t)]$
- the variance function $v(t) = \mathrm{V}[X(t)]$
- the (auto)covariance function $r(s,t) = C(X(s), X(t))$
- the (auto)correlation function $\rho(s,t) = r(s,t)/\sqrt{v(s)v(t)}$

if these functions exist finitely.

In many applications time series data tend to stay around an average level. In this case we model a time series as a sequence of dependent random variables with constant mean function

$$m(n) = \mathrm{E}[X_n] = m, \quad m(t) = \mathrm{E}[X_t] = m,$$

fixed for all m or t.

Weakly stationary A stochastic process $\{X(t)\}$ is said to be weakly stationary if

- the mean value function $m(t) \equiv m$ is constant
- the covariance function $r(s,t) = r(t-s)$ is a function of the time difference $t-s$ only.

Thus a weakly stationary random process describes a random entity which fluctuates around a constant average and whose covariance of any two values depends only on the time distance between them and not on "clock time".

2 Series and parallel systems

Basic case ("snap-shot observation") Two, independent components each working with probability p

function probability:

$$R_{ser}(p) = \mathrm{P}(\text{2-series system works}) = p^2$$
$$R_{par}(p) = \mathrm{P}(\text{2-parallel system works}) = 1-(1-p)^2$$

Extended case ("life-length of system") n components with independent, exponentially distributed life-lengths $T_1,\ldots,T_n$ characterized by intensities $\lambda_i, 1 \le i \le n$. We are interested in the system time, i.e. the life-length of the system

$$T_{ser} = \min(T_1,\ldots,T_n) \qquad T_{par} = \max(T_1,\ldots,T_n).$$

The corresponding *survival functions* are

$$\begin{aligned} R_{ser}(t) &= \mathrm{P}(\text{series system functions at time } t) = \mathrm{P}(T_{ser} > t) \\ &= e^{-(\lambda_1+\ldots+\lambda_n)t} \\ R_{par}(t) &= \mathrm{P}(\text{parallel system functions at time } t) = \mathrm{P}(T_{par} > t) \\ &= 1-(1-e^{-\lambda_1 t})\cdots(1-e^{-\lambda_n t}). \end{aligned}$$

Consider Example 2.11 and 2.12 in the lecture notes IK for derivations from this model!

3 The Poisson process

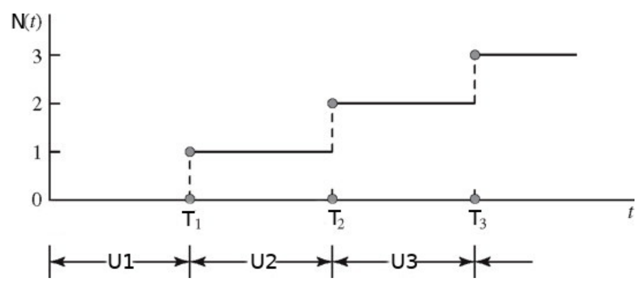
The Poisson process is the continuous time limit of the Bernoulli process. This is discussed in IK 2.4. and 2.5..

The Poisson process is used to model events that occur randomly and independently in time (for example in insurance industry to model rare events that carry a high risk, e.g. a flood disaster or an air crash). There are several equivalent definitions of the process:

Definition I Let $\{U_k, 1 \leq k < \infty\}$ be a sequence of independent and identically distributed random variables all having the exponential distribution with intensity $\lambda > 0$, $\mathrm{P}(U_i \leq t) = 1 - e^{-\lambda t}, t \geq 0$. Let $T_k = \sum_{i=1}^{k} U_i, k \geq 1$, denote the corresponding sequence of accumulated sums. The stochastic process $\{N(t), t \geq 0\}$ in continuous time and with nonnegative integer values that counts the number of time points $\{T_k, k \geq 1\}$ that occurs before time t,

$$N(t) = \sum_{k=1}^{\infty} 1_{\{T_k \leq t\}}, \quad t \geq 0,$$

is called the Poisson process with intensity λ.



Considering the random times T_k as the time epochs of the occurrences of "Poisson events" and the random variables U_k their inter-occurrence times, then

$$N(t) = \text{ number of Poisson events in the interval } [0, t].$$

So the Poisson process is a jump process $\{N(t)\}$ in continuous time which starts from $N(0) = 0$ and steps from each integer value to the next larger value in such a way that the waiting times between the jumps are independent and exponential with mean $1/\lambda$. The distribution of the random process $t \mapsto N(t)$ at a fixed time t is Poisson: $N(t) \in \mathrm{Po}(\lambda t)$ and especially $\mathrm{E}[N(t)] = \lambda t$.
With the help of the equality $\{N(t) \geq n\} = \{T_n \leq t\}$ one can show the equivalence of definition I to:

Definition II The Poisson process $\{N(t), t \geq 0\}, N(0) = 0$, is an integer-valued stochastic process such that

- the successive increments $N(t_1), N(t_2) - N(t_1), \ldots, N(t_k) - N(t_{k-1})$ are independent random variables for any choice of $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k$ and $k \geq 1$; and
- for some $\lambda > 0$: $N(t) - N(s) \in \text{Po}(\lambda(t-s))$ for any $s \leq t$.

A natural parameter interpretation is $\lambda = \text{E}[N(1)]$= expected number of Poisson events per unit time.

Superposition property If $\{N_1(t)\}$ and $\{N_2(t)\}$ are two independent Poisson processes with respective intensities λ_1 and λ_2, then the sum $M(t) = N_1(t) + N_2(t)$ defines a new Poisson process which has intensity $\lambda_1 + \lambda_2$.

The third definition of the Poisson process emphasizes the interpretation of λ as an infinitesimal jump intensity:

Definition III The Poisson process $\{N(t), t \geq 0\}, N(0) = 0$, is an integer-valued stochastic process such that

- the successive increments $N(t_1), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$ are independent,
- as $h \to 0, \text{P}(N(t+h) - N(t) = 1) = \lambda h + o(h)$,
- as $h \to 0, \text{P}(N(t+h) - N(t) \geq 2) = o(h)$.

Spatial Poisson process A spatial point process is a collection of random points in a spatial region S of the plane or in all of $\mathbb{R}^2$. We write $N(A)$ for the number of points in a set $A \subset S$, and allow arbitrary (but well-defined) sets A. Denote by $|A|$ the volume of the set. We say that $\{N(A), A \subset S\}$ is a spatial Poisson process with intensity $\lambda > 0$ if

- for each set A, $N(A)$ has the Poisson distribution with expected value $\lambda|A|$,
- for any pair of disjoint sets A and B in S, the random variables $N(A)$ and $N(B)$ are independent.