UPPSALA UNIVERSITET
**Matematiska Institutionen**
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VT22
Assignment 1
2022/03/30

**Partial Differential Equations with Applications to Finance**

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**Instructions:** There are two problems each worths 10 points. A score of 12 points yield 1 bonus point in the final grade. Your answer should be well motivated in order to receive full credit in each question.

Please submit your solutions under "Assignment 1" before midnight 19th April. The submission can either be handwritten or typed, but it should be in .pdf format. You should expect 5 working days for your submission to be corrected.

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**1.** *(The Ornstein-Uhlenbeck process)*
For a given standard Brownian motion $W$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the Ornstein-Uhlenbeck process which solves the following SDE:

$$dX_t = \mu X_t dt + \sigma dW_t, \quad X_0 = x, \tag{1}$$

where $\mu, \sigma \in \mathbb{R}$.

i) Show that equation 1 admits a unique strong solution, and use Ito's formula to find the solution to the above equation *(Hint: consider the process $e^{-\mu t}X_t$).*

ii) Fix $T \geq 0$, find $\mathbb{E}[X_T]$ and $Var[X_T]$.

iii) We wish now to compute the characteristic function $\phi$ of $X_T$:

$$\phi_{X_T}(\xi) := \mathbb{E}[e^{i\xi X_T} | X_0 = x], \text{ for } \xi \in \mathbb{R}.$$

Fix $\xi \in \mathbb{R}$, use the Feynman-Kac theorem to show that the function $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ defined by $u(t, x) := \mathbb{E}[e^{i\xi X_T} | X_t = x]$ satisfy the following PDE:

$$u_t + \mu x u_x + \frac{1}{2}\sigma^2 u_{xx} = 0, \text{ for all } (t, x) \in [0, T) \times \mathbb{R}.$$

Determine the terminal condition $u(T, x)$.

iv) Propose a suitable terminal condition for iii), and use the ansatz

$$u(t, x) = \exp(\beta(t) + i\xi\alpha(t)x)$$

for some functions $\alpha(\cdot), \beta(\cdot)$ to prove the identity

$$\phi_{X_T}(\xi) = \exp\left(i\xi x e^{\mu T} - \frac{\sigma^2 \xi^2}{4\mu}(e^{2\mu T} - 1)\right).$$

v) Recall that we can determine the $k-$th moment of a random variable $Y$ by computing the $k-$th derivative of its characteristic function $\phi_Y(\xi)$:

$$\mathbb{E}[Y^k] = i^{-k}\phi_Y^{(k)}(0).$$

Use the identity in iv) to check you result in ii).

**2.** i) Let $W = (W_1, \dots, W_n)$ be a standard $n-$dimensional Brownian motion. Let $(c_1, \dots, c_n) \in \mathbb{R}^n$ and let $\alpha > 0$ be a constant. Consider the process $Z(t) \in \mathbb{R}$:

$$Z(t) = \sum_{j=1}^{n} c_j W_j(\alpha t), \ \ t \geq 0.$$

Show that $Z$ is a standard Brownian motion i.f.f. $\frac{1}{\alpha} = \sum_{j=1}^{n} c_j^2$.

ii) Consider the $n-$dimensional hypercube:

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_1| < 1, \dots, |x_n| < 1\}.$$

Let $W$ be an $n-$dimensional BM with starting point $W_0 = (\frac{1}{2}, \dots, \frac{1}{2}) \in H$. Define the exit time of $W$ from $H$ as $\tau_H$.

(a) Use Dynkin's formula to show that $\mathbb{E}[\tau_H] < 1$.

(b) Formulate a suitable Poisson problem and show that $\mathbb{E}[\tau_H] < 1$.