Lecture 1. Probability Preliminaries

· Definition 1.1 (Probability space)

A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where

- Ω: set of elemetary outcomes ω.
- $\mathcal{F}$: a σ-algebra (i.e. $\Omega \in \mathcal{F}$, closed under complements, countable unions)

  (set of events, collection of subsets of Ω)

$\Rightarrow$ $(\Omega, \mathcal{F})$: mble space.

$(\Omega, \mathcal{F}, \mu)$: measure space.

If μ is a prob. measure $\Rightarrow$ prob space.

- $\mathbb{P}$: a probability measure: $\mathcal{F} \to [0,1]$. $\mathbb{P}(\Omega) = 1$.

  (Kolmogorov axioms).

Ex. 1.1 (Toss 2 coins.) $\to$ Countable

$\Omega = \{HH, HT, TH, TT\}$, $|\Omega| = 4$.

$\mathcal{F} \overset{e.g.}{=} \{\{HH, HT\}, \{TH, TT\}, \Omega, \emptyset\}$.

$\mathbb{P}$ = e.g. fair.

Ex. 1.2 (Toss infinitely many coins) $\to$ Uncountable.

$\Omega = \{H, T\}^{\infty}$. / each single outcome is an infinite binary string.

$\mathcal{F}$ = (e.g) "First is H" = $\{\{HH\ldots\}, \{HT\ldots\}, \ldots\}$.

$\mathbb{P}$ = e.g. $\mathbb{P}(A) = \frac{1}{2}$.

Def. 1.1.2 (Random variable)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, a random variable

$X : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-mble function, i.e.

$\omega \to X(\omega)$

$$X^{-1}(A) := \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F},$$

for every Borel set A.

Remark. A r.v. induces a prob-measure:

$$\mathbb{P}_X(\underset{\in\mathbb{R}}{A}) = \mathbb{P}(\underset{\in\Omega}{X^{-1}(A)}) =: \mathbb{P}(X \in A)$$

[One can get a new probability space : $(\Omega, \mathcal{F}, \mathbb{P}_X)$.]

$\mathbb{P}_X$: distribution of X. , $F_X(x) := \mathbb{P}_X((-\infty, x])$

We say that two r.v's $X \overset{a.s.}{=} Y$ if

$$\mathbb{P}(\{\omega \in \Omega : X(\omega) \neq Y(\omega)\}) = 0.$$

(Note that "a.s." is stronger than "d".

$X \overset{d}{=} Y$ if $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all A.)

Notation: $\{X = x\} = \{\omega \in \Omega : X(\omega) = x\}$.

Def 1.1.3 (Expectation).

On $(\Omega, \mathcal{F}, \mathbb{P})$, the expectation of X is defined as

- $$\mathbb{E}[X] := \int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \int_{\mathbb{R}} x \, d\mathbb{P}_X(x)$$

$$\underset{\text{notation}}{=} \int X \, d\mathbb{P}.$$

- Let $f: \mathbb{R} \to \mathbb{R}$ be Borel-mble, integrable, then

$$\mathbb{E}[f(X)] := \int_\Omega f(X(\omega))\, d\mathbb{P}(\omega) = \int_{\mathbb{R}} f(x)\, d\mathbb{P}_X(x).$$

- The conditional expectation of X given event A is

$$\mathbb{E}[X \mid A] = \frac{\mathbb{E}[X 1_A]}{\mathbb{P}(A)}.$$

e.g. $\mathbb{E}[X \mid Y = y_i]$: conditional expectation on the event $\{\omega \in \Omega : Y(\omega) = y_i\}$.

$$\mathbb{E}[X \mid \mathcal{G}] = \sum_{k=1}^{n} \mathbb{E}[X \mid A_k] 1_{A_k}, \quad \sigma(A_1, \ldots, A_n) = \mathcal{G}.$$

$\hookrightarrow \mathcal{G} \subset \mathcal{F}$.

- Let $\mathcal{G} \subset \mathcal{F}$. Conditional expectation $\mathbb{E}[X \mid \mathcal{G}]$ is the unique

fcn $: \Omega \to \mathbb{R}$ s.t.

i) $\mathbb{E}[X \mid \mathcal{G}]$ is $\mathcal{G}$-mble

ii) $\int_A \mathbb{E}[X \mid \mathcal{G}]\, d\mathbb{P} = \int_A X \, d\mathbb{P}$ for all $A \in \mathcal{G}$.

• Properties of conditional expectation

i). $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[X]$.

ii) If X is $\mathcal{G}$-mble. $\mathbb{E}[X \mid \mathcal{G}] = X$.

iii) If $X \perp\!\!\!\perp \mathcal{G}$. $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$.

Def 1.1.4 (Stochastic Process).

A. S.P. is a collection of r.v.s. indexed by $t \in T$.

In this course, we consider $T = [0, \infty)$.

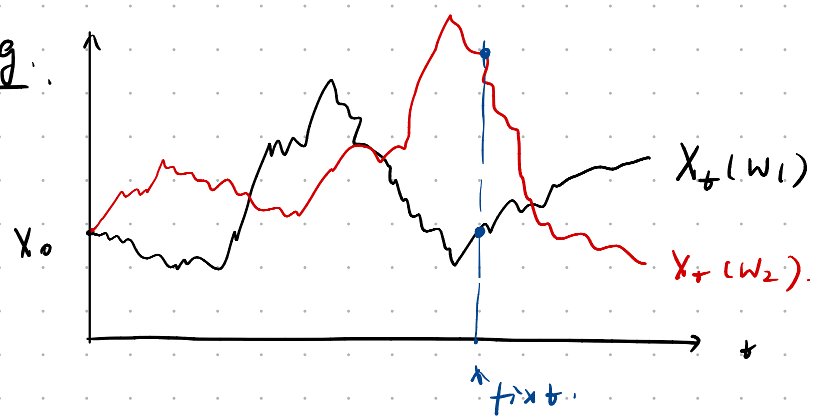
• For each t, we have a r.v.

$$\omega \to X_t(\omega), \quad \omega \in \Omega.$$

• For each ω, we have a trajectory

$$t \to X_t(\omega), \quad t \in T$$

e.g.



Alternatively, X can also be seen as a map:

$$X : T \times \Omega \to \mathbb{R}$$

<u>Notations</u>, $X(t)$, X_t, $X_t(\omega)$, $X(t,\omega)$, ...

- <u>Def 1.1.5. (Brownian motion)</u>

A stochastic process W is called a B.M or Wiener process if

i) $W(0)=0$

ii) W has continuous trajectories.

iii) W has independent increments.

(i.e. if $t_1 < t_2 < t_3 < t_4$, $W(t_4) - W(t_3) \perp\!\!\!\perp W(t_2) - W(t_1)$)

iv). Increments are Gaussian, and

if $s<t$, $W(t) - W(s) \sim N(0, t-s)$

↳ variance.

- A n-dim BM is $W = (W_1, \dots, W_n)$. where $W_1, \dots, W_n$ are mutually independent.

<u>Def</u> 1.1.6. (Filtration).

(problem: we never know what ω we drawn, only what happened up to now)

A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is a family of increasing sub-σ-algebras of $\mathcal{F}$, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$. then we write $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ as a filtered probability space.

- A Y/N question that can be answered at time t can be answered at any later time.

E.x. 1.1 again.

$$\Omega = \{HH, HT, TH, TT\}$$

Let X_i = result of the ith toss. $\in \{0, 1\}$.

Assume "First is Head".

which is our ω?

$$X_1\{HH\} = X_1\{HT\} = 1, \quad X_1(TH) = X_1(TT) = 1.$$

$$\mathcal{F}_1 = \sigma(X_1) = \{\{HH, HT\}, \{TH, TT\}, \Omega, \emptyset\}.$$

Denote the filtration generated by X up to time t. by $\mathcal{F}_t^X$.

"The information of X up to t".

- If by observing X up to t $(\mathcal{F}_t^X)$, we can determine if $A \in \mathcal{F}$ has occured or not, we say $A \in \mathcal{F}_t^X$

- Let Z be a r.v. and we can determine the value of Z by $\mathcal{F}_t^X$.

we say Z is $\mathcal{F}_t^X$ – mble. $Z \in \mathcal{F}_t^X$.

- If X, Y are s.p. and $Y_t \in \mathcal{F}_t^X$ for all $t \geq 0$. we say Y is adapted to $\mathcal{F}^X$; $Y \in \mathcal{F}^X$.

e.g. • $Y_t = \sup_{0 \leq s \leq t} X_s$. $\in \mathcal{F}_t^X$

• $Y_t = \sup_{0 \leq s \leq 3t} X_s$ $\notin \mathcal{F}_t^X$

↳ need to look into the future.

Def 1.1.7 (Stopping time).

Let $\{\mathcal{F}_t\}_{t \geq 0}$ be an increasing family of σ-algebras. A fcn $\tau : \Omega \to [0, \infty)$ is called a stopping time w.r.t $\{\mathcal{F}_t\}$. if

$$\{ w : \tau(w) \leq t \} \in \mathcal{F}_t \quad , \text{ for all } t.$$

↳ is a r.v.

- **Remark** TFAE.

i). τ is a $\mathcal{F}_t$ – stopping time, ii). $\mathbb{1}_{[0,\tau]}$ is $\mathcal{F}_t$ – adapted.

Properties If τ_1, τ_2 are $\mathcal{F}_t$-stopping times.

i). $\tau_1 \wedge \tau_2$ is also a $\mathcal{F}_t$-stopping time

ii) $\tau_1 \vee \tau_2$ — " ————.

iii) $\tau_1 + \tau_2$ — " ————.

Exercise 1. Prove the above statements.

Exercise 2. Prove that $\tau_1 - \tau_2$ is NOT a stopping time.

E.X. 1.3 i). Every deterministic t is a stopping time.

$$\{t \le t\} \in \{\emptyset, \Omega\}.$$

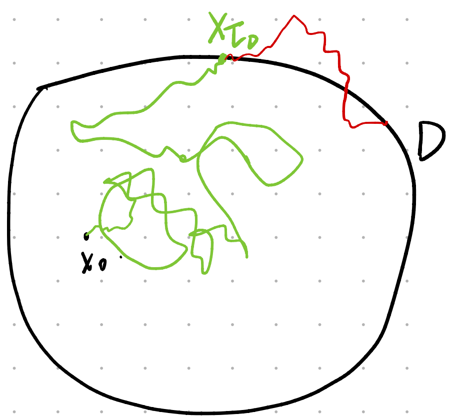
ii). (Hitting times / Exit times).

Let X_t be a s.p. in $\mathbb{R}^n$, Let $X_0 \in D \subset \mathbb{R}^n$, we define the <u>first exit time</u> of X from D as

deterministic.

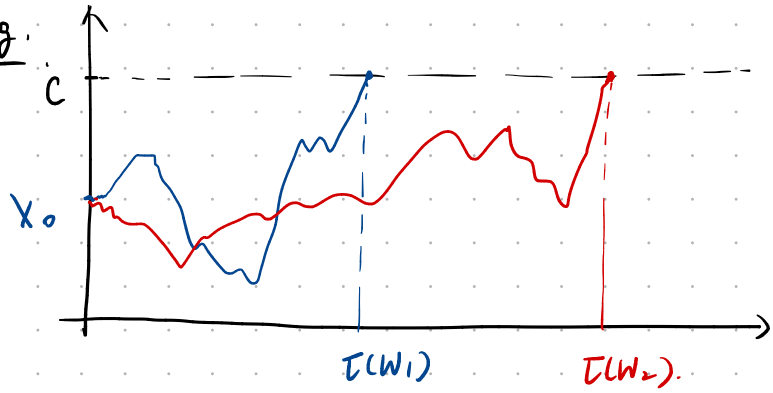
$$\tau_D := \inf\{t > 0 : X(t) \notin D\}.$$

<u>e.g.</u>



Let $A \in \mathbb{R}^n$ be closed and non-empty, the first hitting time of F by X is defined as $\tau_{\mathbb{R}^n \setminus F}$.

e.g.



$\tau(w) := \inf\{t>0 : X_t(w) = c\}$.

e.g. Let $\tau(w) := \sup\{t>0 : X_t(w) = c\}$ is NOT a stp time