Lecture 12. Merton's problem

<u>Last time</u>. HJB eqn $\Longleftrightarrow$ Stochastic optimal control

<u>Today</u> . Examples. (Especially in finance).

<u>Example 10.1</u> (Merton's asset allocation problem)

Consider the BS-world, where we have a risky asset S_t:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

and a risk-free asset B_t:

$$dB_t = r B_t dt$$

where $\mu > r > 0$.

Let's assume we want to invest a fraction α_t in S_t, and the rest $(1-\alpha_t)$ in B_t, (assume $\alpha_t \in L^2$) i.e.

$$dX_t^\alpha = \frac{\alpha_t X_t^\alpha}{S_t} dS_t + \frac{(1-\alpha_t) X_t^\alpha}{B_t} dB_t.$$

$$= (\mu \alpha_t + r(1-\alpha_t)) X_t^\alpha dt + \sigma \alpha_t X_t^\alpha dW_t.$$

We wish to maximise the expected utility at time T:

$$V(t,x) = \sup_\alpha \mathbb{E}_{t,x}[\phi(X_T^\alpha)].$$

<u>Remark</u>: i) The utility fcn ϕ is usually increasing and concave.

ii). We look for α_t, Markov control, s.t.

$\alpha_t \geq 0$ (no borrowing)

Step 1. Write the HJB :

$$\begin{cases} \dfrac{\partial V}{\partial t} + \sup_{\alpha} \{ \mathcal{L}^{\alpha} V(t,x) + 0 \} = 0. \\ V(T,x) = \phi(x) \end{cases}$$

where $\mathcal{L}^{\alpha} V = x(\alpha\mu + (1-\alpha) r) V_x + \frac{1}{2}\sigma^2\alpha^2 x^2 V_{xx}$.

- We consider "power utility": $\phi(x) = x^{\gamma}$, $\gamma \in (0,1)$;
- Use the ansatz: $V(t,x) = \eta(t)\phi(x)$ where $\eta(T) = 1$.

Step 2. solve for α^* :

$$V_t = \eta_t \phi \, , \, V_x = \eta \phi_x \, , \, V_{xx} = \eta \phi_{xx} \, ,$$

$$\sup_{\alpha} \{ \mathcal{L}^{\alpha} V(t,x) \} = \sup_{\alpha} \{ \eta x ((\mu - r)\alpha + r) \phi_x + \tfrac{1}{2}\sigma^2\alpha^2 x^2 \eta \phi_{xx} \}$$

$$= \sup_{\alpha} \{ \eta\gamma((\mu-r)\alpha + r) x^{\gamma} + \tfrac{1}{2}\underbrace{\gamma(\gamma-1)}_{<0}\sigma^2\alpha^2 \eta x^{\gamma} \}.$$

$$\Rightarrow \alpha^* = \frac{\gamma\eta(\mu - r) x^{\gamma}}{-\gamma(\gamma-1)\sigma^2\eta x^{\gamma}} = \frac{\mu - r}{(1-\gamma)\sigma^2}$$

↳ in fact a constant!

check: $\alpha^* \geq 0$? $\checkmark$.

Step 3. plug in α^*, solve PDE.

$$\eta_t x^{\gamma} + \eta x^{\gamma} \underbrace{\{ \gamma((\mu - r)\alpha^* + r) + \tfrac{1}{2}\gamma(\gamma-1)\sigma^2(\alpha^*)^2 \}}_{=: k, \text{ constant}} = 0.$$

$$\Rightarrow \eta_t = -k\eta \, , \, \eta(T) = 1.$$

$$\Rightarrow \eta(t) = \exp\{k(T-t)\}.$$

Therefore, $\boxed{V(t,x) = \exp\{k(T-t)\}\, x^{\gamma}.}$

By the verification thm, we have found the correct V and α^*.

Remark. i) When $\gamma = 1$, $\varphi(x) = x$, everything in S ! $\rightarrow$ risk neutral

ii) If we add "no shortselling" as a constraint, i.e.

$$\alpha(t) \le 1. \quad \text{then} \quad \alpha^* = \frac{\mu - r}{(1-\gamma)\sigma^2} \wedge 1.$$

iii) We can also take $\phi = \log x$ $\rightarrow$ "kelly criterion".

Ansatz: $V(t,x) = \phi(x) + \lambda(t)$. , then similarly.

$$\alpha^* = -\frac{\mu - r}{\sigma^2} \frac{V_x}{x V_{xx}} = \frac{\mu - r}{\sigma^2}$$

$$\lambda_t + \underbrace{\left(r + \frac{(\mu-r)^2}{2\sigma^2}\right)}_{:=K} = 0, \quad \lambda(T) = 0. \qquad \text{(check!)}$$

$$\Rightarrow \lambda(t) = K(T-t).$$

$$V(t,x) = \log(x) + K(T-t).$$

Alternatively, By Itô,

$$\mathbb{E}_{t,x}[\log X_T] = \log x + \mathbb{E}_{t,x}\left[\int_t^T \left((\mu-r)\alpha_s + r - \tfrac{1}{2}\alpha_s^2\sigma^2\right) ds\right]$$

$$\Rightarrow \quad \alpha(t,x) = \frac{\mu - r}{\sigma^2}. \quad \rightarrow \text{Same conclusion!}$$

Exercise. Use this argument to check the result when $\phi = x^\gamma$.

Example 10.2 (Portfolio optimisation)

Two assets, $dS_i = \mu_i S_i dt + \sigma_i S_i dB_t$, $i \in \{1,2\}$. solve

$\sup_\alpha \mathbb{E}_{t,x}[\phi(X_T^\alpha)]$, where α is the proportion in S_1, and $\phi = x^\gamma$, $\gamma \in (0,1)$.

Sol. Entirely parellel to 12.1.

Example 10.3. (Exercise 9.1)

Solve $V(t,x) = \sup_\alpha \mathbb{E}_{t,x}[e^{-r(T-t)} g(X_T^\alpha)]$, where

$$dX_s^\alpha = r X_s^\alpha + \alpha_s X_s^\alpha dB_s.$$

Sol. Let $u(t,x) = \sup_\alpha \mathbb{E}_{t,x}[g(X_T^\alpha)]$.

Note that $V(t,x) = e^{-r(T-t)} u(t,x)$, where

$$\begin{cases} u_t + \sup_\alpha \{ r x u_x + \frac{1}{2}\alpha^2 x^2 u_{xx} \} = 0. \\ u(T,x) = g(x) \end{cases}$$

Alternatively, plug in $u = e^{r(T-t)} v$, we have

$$\begin{cases} V_t + \sup_{\alpha} \{ r x V_x + \frac{1}{2} \alpha^2 x^2 V_{xx} \} - rV = 0. \\ V(T, x) = g(x) \end{cases}$$

→ Check!

Thm 10.4 (HJB with discounting).

Let $v(t,x) = \sup_{\alpha \in A} \mathbb{E}_{t,x}\left[\int_t^T e^{-\int_t^s \beta_u \, du} \psi_s^\alpha \, ds + e^{-\int_t^T \beta_s \, ds} \phi(X_T^\alpha) \right]$,

where $\beta_t = \beta(t, X_t^\alpha, \alpha_t)$. then V solves

$$\boxed{\begin{cases} V_t + \sup_{\alpha \in A} \{ \mathcal{L}^\alpha + \psi^\alpha - \beta(t,x,\alpha) V \} = 0 \\ V(T, x) = \phi(x) \end{cases}}$$

Cor 10.5 (HJB for infinite time horizon).

Consider the optimization problem

$$V(x) = \sup_{\alpha} \mathbb{E}_{0,x}\left[\int_0^\infty e^{-\beta t} \psi(X_t^\alpha, \alpha) \, dt \right]$$

where $\beta > 0$ (discounting), HJB says

$$\boxed{\sup_{\alpha} \{ \mathcal{L}^\alpha V + \psi^\alpha - \beta V \} = 0.}$$

Remark. i) V is time-homogeneous.

ii) There's no boundary condition, but we need

$$e^{-\beta T} V(X_T) \xrightarrow{T \to \infty} 0$$

for the verification thm.

Example 10.6 (Optimal consumption for an immortal)

We now extend 10.1, and consider that we have the opportunity to spend the money continuously. We consider now the infinite time horizon case $T=\infty$.

Let C_t be the rate of consumption and ψ be the running utility fcn.

Want:

$$V(x) = \underbrace{\sup_{\alpha, c}}_{\text{2-dim control}} \mathbb{E}_{0,x}\left[\int_0^\infty e^{-\beta s} \psi(C_s X_s)\, ds \right]$$

where $\beta > 0$. $C_t X_t$ is the amount spent at t. Denote $X_t^{\alpha,c}$ by X_t, then

$$dX_t = (\underbrace{\alpha X_t \mu}_{\text{risky}} + \underbrace{(1-\alpha) X_t r}_{\text{risk-free}})\, dt + \alpha X_t \sigma\, dB_t - \underbrace{C_t X_t\, dt}_{\text{cash spent}}.$$

Correspondingly, $\mathcal{L}^{\alpha,c}$ becomes:

$$\mathcal{L}^{\alpha,c} f = (\alpha\mu + (1-\alpha) r - c)\, x f_x + \frac{1}{2}\sigma^2 \alpha^2 x^2 f_{xx}$$

- Consider again, $\psi(x) = x^{\gamma}$, $\gamma \in (0,1)$.

Step 1. Write HJB.

$$\sup_{\alpha, c} \left[\mathcal{L}^{\alpha,c} V + \psi(cx) - \beta V \right] \overset{(*)}{=} 0$$

Plug in $\mathcal{L}^{\alpha,c}$, we get:

$$(*) = \sup_{\alpha, c} \{ \underbrace{(\alpha\mu + (1-\alpha) r) x V_x + \frac{1}{2}\sigma^2\alpha^2 x^2 V_{xx}}_{\text{depends only on } \alpha} + \underbrace{\psi(cx) - c x V_x}_{\text{only on } c} \}.$$

$$= \sup_{\alpha}\left\{ (\alpha\mu + (1-\alpha) r) x V_x + \frac{1}{2}\sigma^2\alpha^2 x^2 V_{xx} \right\} + \sup_{c}\left\{ \Psi(cx) - cx V_x \right\}.$$

Step 2. Solve α^*, c^*.

Let $V(x) = D \cdot \Psi(x)$

$$(*) = \sup_{\alpha}\left\{ D\gamma(\alpha\mu + (1-\alpha)r) x^{\gamma} + \frac{1}{2}\gamma(\gamma-1) D\sigma^2\alpha^2 x^{\gamma} \right\} + \sup_{c}\left\{ (cx)^{\gamma} - \gamma c D x^{\gamma} \right\}.$$

$$\Rightarrow \alpha^* = \frac{\mu - r}{(1-\gamma)\sigma^2}, \quad c^* = \underset{>0}{\underbrace{D}}^{\frac{1}{\gamma-1}}. \quad \Rightarrow \text{both nonnegative.}$$

Step 3. Solve PDE.

plug in α^*, c^*:

$$D x^{\gamma}\Big(\underbrace{\gamma(\alpha^*\mu + (1-\alpha^*)r) + \frac{1}{2}\gamma(\gamma-1)\sigma^2(\alpha^*)^2}_{:= k,\ \text{const.}}\Big) + x^{\gamma}\big((\underset{D^{\frac{1}{\gamma-1}}}{\underbrace{c^*}})^{\gamma} - \gamma c^* D\big) - \beta D x^{\gamma} = 0$$

$$\Rightarrow \quad (k-\beta) D + (1-\gamma)\cdot D^{\frac{\gamma}{\gamma-1}} = 0$$

$$\Rightarrow \quad D = \left(\frac{1-\gamma}{\beta - k}\right)^{1-\gamma}.$$

Therefore, $V(x) = \left(\frac{1-\gamma}{\beta-k}\right)^{1-\gamma} x^{\gamma}$; for β sufficiently large, i.e., the verification thm holds. when $\beta > k = \gamma r + \frac{1}{2}\gamma\frac{(\mu-r)^2}{(1-\gamma)\sigma^2}$. Furthermore,

α^*, c^* are constant processes as defined.

Comments on HJB. 1. α^* might not exist.

Example 10.7 (Unattainable optimiser / 9.2).

Let $dX_t = \alpha_t dt + dB_t$. Let $V(t,x) = \inf_{\alpha \in A} \mathbb{E}_{t,x}[X_T^2]$.

i.e. Bring X_T as close as possible to 0.

In this case the optimal control does not exist!

- To see this, take $\alpha_t = -c X_t$, X_t becomes Ornstein-Uhlenbeck.

$$X_T = e^{-c(T-t)} x + \int_t^T e^{-c(T-r)} dW_r$$

$$J^{-cx}(t,x) = \mathbb{E}[X_T^2] = x^2 e^{-2c(T-t)} + \int_t^T e^{-2c(T-r)} dr$$

Itô isometry

$$= \left(x^2 - \frac{1}{2c}\right) e^{-2c(T-t)} + \frac{1}{2c}.$$

Obs. that $V(t,x) \geq 0$, and $V(t,x) \leq \inf_c J^{-cx}(t,x)$, then

$$0 \leq V(t,x) \leq \inf_c \left\{ \left(x^2 - \frac{1}{2c}\right) e^{-2c(T-t)} + \frac{1}{2c} \right\}$$

$$\leq \lim_{c \to \infty} \left\{ \left(x^2 - \frac{1}{2c}\right) e^{-2c(T-t)} + \frac{1}{2c} \right\}$$

$$= 0.$$

- Therefore, $V(t,x) = 0$ for all $t < T$, $x \in \mathbb{R}$.

- But α^* does not exist:

Assume α^* exists, by Itô,

$$d(X_t^*)^2 = 2X_t^*\alpha^* dt + 2X_t^* dB_t + dt$$

$$0 = \mathbb{E}[(X_T^*)^2] = x^2 + \mathbb{E}\left[\int_t^T (2X_s^*\alpha^* + 1)\, ds\right] + \underbrace{\mathbb{E}\left[\int_t^T 2X_s^*\, dB_s\right]}_{0}.$$

Taking $t \to T$,

$$-x^2 = \liminf_{t\to T} \mathbb{E}\left[\int_t^T (2X_s^*\alpha^* + 1)\, ds\right]$$

$$\geq \mathbb{E}\left[\liminf_{t\to T} \int_t^T (2X_s^*\alpha^* + 1)\, ds\right]$$

Fatou

$$= 0.$$

It can't hold that $x^2 \leq 0$ for all $x \in \mathbb{R}$, contradiction. α^* doesn't exist.

- Alternatively, the associated HJB is

$$\begin{cases} V_t + \frac{1}{2} V_{xx} + \inf_{\alpha \in \mathbb{R}} \{\alpha V_x\} = 0. \\ V(T, x) = x^2 \end{cases}$$

when $V_x \neq 0$. $\alpha = \pm\infty$. when $V_x = 0$. α is not defined. In either way α^* cannot be attained.

2. α^* is not necessarily unique.

Trivial.