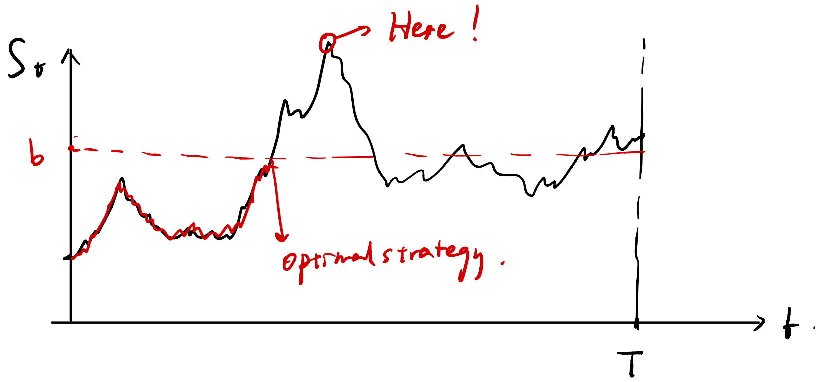
Lecture 11, Optimal Stopping and Free-bdr Problems

Let $T > 0$, assume we hold a stock S_t, when do we sell it, before T?



Problem: we can't tell if it's the highest point.

$$\{ S_t = \max_{0 \le r \le T} S_r \} \notin \mathcal{F}_t .$$

More realistically: set a barrier b, once $S_t \geq b$, sell.

Let $T > 0$ and X_t be a stochastic process, an optimal stopping problem is to maximise $\mathbb{E}_{t,x}[g(X_\tau)]$ over some sets of stopping times, where g is called the gain function. Write

$$V(t,x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{t,x}[g(X_\tau)] = \mathbb{E}_{t,x}[g(X_{\tau^*})]$$

value fcn. time-dependent!

set of admissible τ's.

optimal strategy (if exists).

Goal: i) Find V. — How much do we get.

ii) Find τ^*. — When do we stop.

We consider: i) $T = \infty$, so V is time-independent.

ii) X_t is a time-homogeneous Itô diffusion:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t.$$

Some applications

i). Stochastic analysis

Let B_t be a 1D-BM. $B_0 = x$. We consider

$$V(x) = \sup_{\tau} \mathbb{E}_x\left[e^{-r\tau} f(B_\tau)\right].$$

Consider e.g. $f(x) = x^2$ (HW).

Can't wait forever, when $|B|$ is big enough we stop.

ii). Sequential analysis

Suppose we observe a BM with drift:

$$X_t = \mu t + B_t.$$

where μ is a Bernoulli r.v. $\mathbb{P}(\mu = \mu_0) = p$, $\mathbb{P}(\mu = \mu_1) = 1-p$.

Then to make inference, we test

$$H_0 : \mu = \mu_0 \,, \quad H_1 : \mu = \mu_1.$$

We want to minimize the probability of making an mistake,

$\Rightarrow$ Formulate an optimal stopping problem, can be solved explicitly.

(Shiryaev, 1969). inf problem.

iii) Mathematical finance.

American put option: Can choose to exercise at any time before expiration T, and get $g(S_\tau) = (K - S_\tau)^+$.

τ ↳ exercising time.

Want: maximize the discounted payoff, thus the price at time t should be.

$$V(t,x) = \sup_{\tau \in [0,T]} \mathbb{E}^{Q}_{t,x}\left[e^{-r(\tau-t)} (K - S_\tau)^+ \right].$$

we will solve this next time! (w. $T=\infty$).

How to find V and τ^*?

Example 11.1 (When do we sell the stock?).

Let X_t be a GBM. $dX_t = \mu X_t\, dt + \sigma X_t\, dB_t$, $X_0 = x$.

Assume i) Trading is not free: transaction cost $c > 0$.

ii) Rate of discounting $\beta > \mu$: risk-free, tax, inflation ...

Want. Identify a stopping time τ^*. s.t.

$$\mathbb{E}_{0,x}\left[e^{-\beta\tau}(X_\tau - c) \right] \text{ is maximised.}$$

($T=\infty$, starting at $t=0$ ⟹ value fcn is time-homogeneous!)

$$V(x) = \sup_{\tau} \mathbb{E}_{0,x}\left[e^{-\beta\tau}(X_\tau - a)\right]$$

Intuition. There must be some threshold $b > 0$. s.t.

If $x \geq b$, sell immediately $\rightarrow \tau^* = 0$

$b > c$, why?

If $x < b$, wait.

$\hookrightarrow$ until when?

- Let $\tau^* := \inf\{t \geq 0 : X_t \geq b\}$, then this first exit time should be the optimal stopping time.

- What is the boundary b? Unknown. "Free-boundary problem".

Question Can V be characterised by a PDE?

i) When $x \geq b$, $\tau^* = 0$, get immediate payoff.

$$V(x) = x - c.$$

ii) when $x < b$. Since X continuous, X_t will stay below b for $t \in (0, h)$, for h small. Therefore, at time 0:

$$V(x) = e^{-\beta h}\, \mathbb{E}_{0,x}[\underbrace{V(X_h)}_{\text{r.v.}}]$$

By Feynman-Kac, V "should" solve.

$$\begin{cases} \underbrace{V_t}_{=0} + \mathcal{L}V - \beta V = 0, & \text{on } (0,b) \\ V(b) = b - c. \end{cases}$$

discounting.

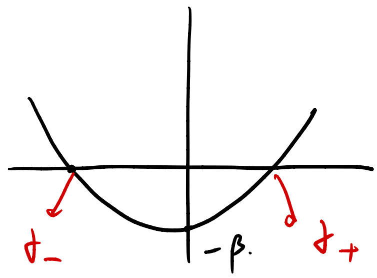
This yields.

$$\begin{cases} \mu x V_x + \frac{1}{2}\sigma^2 x^2 \underbrace{V_{xx}}_{\text{unknown}} - \beta V = 0, & \text{on } (0, \underbrace{b}_{\text{unknown}}). \\ V(b) = b - c. & \\ V(x) = x - c, & \text{on } [b, \infty). \end{cases}$$

<u>Ansatz</u>. $V(x) = x^{\gamma}$. seems useful. plug it in:

$$\mu \gamma \cancel{x^{\gamma}} + \frac{1}{2}\gamma(\gamma-1)\sigma^2 \cancel{x^{\gamma}} - \beta \cancel{x^{\gamma}} = 0.$$

$$\Rightarrow \underbrace{\frac{1}{2}\sigma^2}_{>0}\gamma^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\gamma \underbrace{-\beta}_{<0} = 0$$



Denote the roots by $\gamma_- < 0 < \gamma_+$:

$$\gamma_{\pm} = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) \pm \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\beta\sigma^2}}{\sigma^2}$$

A general solution is then:

$$V(x) = C_1 x^{\gamma_+} + C_2 x^{\gamma_-}$$

However, as $x \to 0$. $V(x) \to 0$. $\Rightarrow C_2 = 0$.

$\Rightarrow V(x) = C_1 x^{\gamma_+}$. for $x \in (0, b)$.

At b: $V(b) = b - c$. $\Rightarrow C_1 = \dfrac{b-c}{b^{\gamma_+}}$

$$\Rightarrow V(x) = \begin{cases} \dfrac{b-c}{b^{\gamma_+}} x^{\gamma_+}, & x \in (0, b). \\ b - c, & x \geq b. \end{cases}$$

problem: b is still unknown!

Recall that we used Feynman-kac on V, implicitly, $V \in C'$.

Therefore, at the free-boundary b,

$$\underbrace{\frac{d}{dx}\left(\frac{b-c}{b^{\gamma_+}} x^{\gamma_+}\right)}_{\text{derivative from the left}}\Big|_{x=b} = \underbrace{1}_{\text{from the right: } x-c}.$$

"Principle of smooth fit".

$$\Rightarrow b = \frac{c\,\gamma_+}{\gamma_+-1}.$$

exercise.

Sanity check: $\gamma_+ - 1 > 0. \iff \beta > \mu.$ ✓.

Remark. i) We wait on $C := (0,b)$ One can verify $V > g$, i.e. waiting gives you higher value. ↳ exercise

We stop on $D := [b,\infty)$, clearly $V = g$. waiting doesn't do you any better.

We define:

Continuation set $C = \{ x \in \mathbb{R} : V(x) > g(x) \}$.

Stopping set $D = \{ x \in \mathbb{R} : V(x) = g(x) \}$.

ii). Typically, we consider $\tau^* := \inf\{ t \geq 0 : X_t \in D \}$.

Is our guess correct? We usually need to construct and prove a verification thm for the problem we study.

Theorem 11.2. (Verification for OS).

Consider the problem $V(x) = \sup_{\tau} \mathbb{E}[g(X_\tau)]$. Assume $\hat{V} \in C^2$, and $\hat{V}$ solves the free boundary problem.

$$\begin{cases} \mathcal{L}\hat{V} \le 0. \\ \hat{V} \ge G \end{cases} \qquad (*)$$

for all x, and there exists C and s.t.

$$\hat{V} > g \text{ in } C, \qquad \hat{V} = g \text{ in } D.$$

then $\hat{V} \equiv V$. and $\tau^* := \inf\{t \ge 0 : X_t \in D\}$ is an optimal strategy.

Remark. i) Moreover, assume g smooth in a neibourhood of ∂C and ∂C is "nice", then $(*)$ splits into:

$$\begin{cases} \mathcal{L}\hat{V} = 0, \text{ in } C. \\ \hat{V} = g, \text{ in } D \\ \hat{V}_x|_{\partial C} = g_x|_{\partial C}. \quad (\text{"Smooth fit"}). \end{cases}$$

ii) In our cases, usually its possible to solve it explicitly by "guess and verify".

Prf. (Sketch) Let $\theta = \tau \wedge T$, Apply Itô on $\hat{V}$:

$$\hat{V}(X_\theta) = \hat{V}(x) + \int_0^\theta \mathcal{L}\hat{V}(X_s)\, ds + \int_0^\theta \cdots dB_s.$$

Since $\mathcal{L}\hat{V} \le 0$,

$$\mathbb{E}[\hat{V}(X_\theta) - \hat{V}(x)] \le 0 + \underbrace{\mathbb{E}[\int_0^\theta \cdots dB_s]}_{0} = 0.$$

$$\Rightarrow \hat{V}(x) \geq \mathbb{E}[\hat{V}(X_\theta)] \underset{\substack{\uparrow \\ \hat{V} \geq g.}}{\geq} \mathbb{E}[\hat{g}(X_\theta)]$$

If $\tau < \infty$. a.s. take T up to ∞: $\lim_{T \to \infty} \tau \wedge T = \tau$.

$$\Rightarrow \hat{V}(x) \geq \mathbb{E}[g(X_\tau)] \quad \text{for all } \tau < \infty. \text{ a.s.}$$

$$\Rightarrow \hat{V}(x) \geq \sup_\tau \mathbb{E}[g(X_\tau)].$$

The other direction: take $\tau = \tau^*$.

$$\mathbb{E}[\hat{V}(X_{\tau^*})] = \hat{V}(x) + \mathbb{E}[\int_0^{\tau^*} (\mathcal{L}\hat{V})(X_s)\, ds] = \hat{V}(x).$$

$\longrightarrow 0$ in C.

$$\hat{V}(x) = \mathbb{E}[\hat{V}(X_{\tau^*})] \underset{\substack{\uparrow \\ \hat{V} = g \text{ in } D}}{=} \mathbb{E}[g(X_{\tau^*})] \le \sup_\tau \mathbb{E}[g(X_\tau)].$$

It follows that τ^* is an optimal strategy. $\boxed{}$

Remark. Other formulations of OS:

i). With discounting.

$$V(x) = \sup_\tau \mathbb{E}_x\left[e^{-\lambda_\tau} g(X_\tau) + \int_0^\tau e^{-\lambda_s} h(X_s)\, ds\right]$$

where the discounting process $\lambda_t \in \mathbb{R}_+$ is

$$\Lambda_t = \int_0^t \lambda(X_s)\,ds.$$

then V solves.

$$\begin{cases} \mathcal{L}V(x) + h(x) - \lambda(x)V(x) = 0 & \text{in } C \\ V = g & \text{in } D \\ \text{smooth fit} & \end{cases}$$

(check with II, I).

ii). The "inf" equivalence.

Sometimes we want to minimize the cost, the cost function

$$V(x) = \inf_{\tau} \mathbb{E}[g(X_\tau)].$$

the method of "guess and verify" is entirely parallel

Applications $\rightleftarrows$ optimal stopping $\rightleftarrows$ Free-bdr problems