Lecture 12. American Options.

Last time. Basic optimal stopping theory $\rightleftarrows$ free bdr problems.

Today. Examples.

Example 12.1 (Perpetual American put option).

We wish to find the optimal exercising time of an American put with strike $K>0$, and with the underlying described in the risk-neutral setting by.

$$dX_t = (r-\delta) X_t \, dt + \sigma \, dB_t.$$

where $r>0$ is the risk-free rate, $\delta \in [0,r)$ is the dividend.

We can exercise it at anytime $t \geq 0$, it never expires.

payoff at t : $(K-X_t)^+$.

Step 1. Write down the value function, C, D and τ^*.

$$V(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-r\tau} (K - X_\tau)^+ \right].$$

dividend is hidden here

risk-neutral pricing

"fair price".

• What does the continuation region C look like?

"when it's good enough, stop".

when $x \geq K$. should always wait! $\Rightarrow [K, \infty) \subset C$.

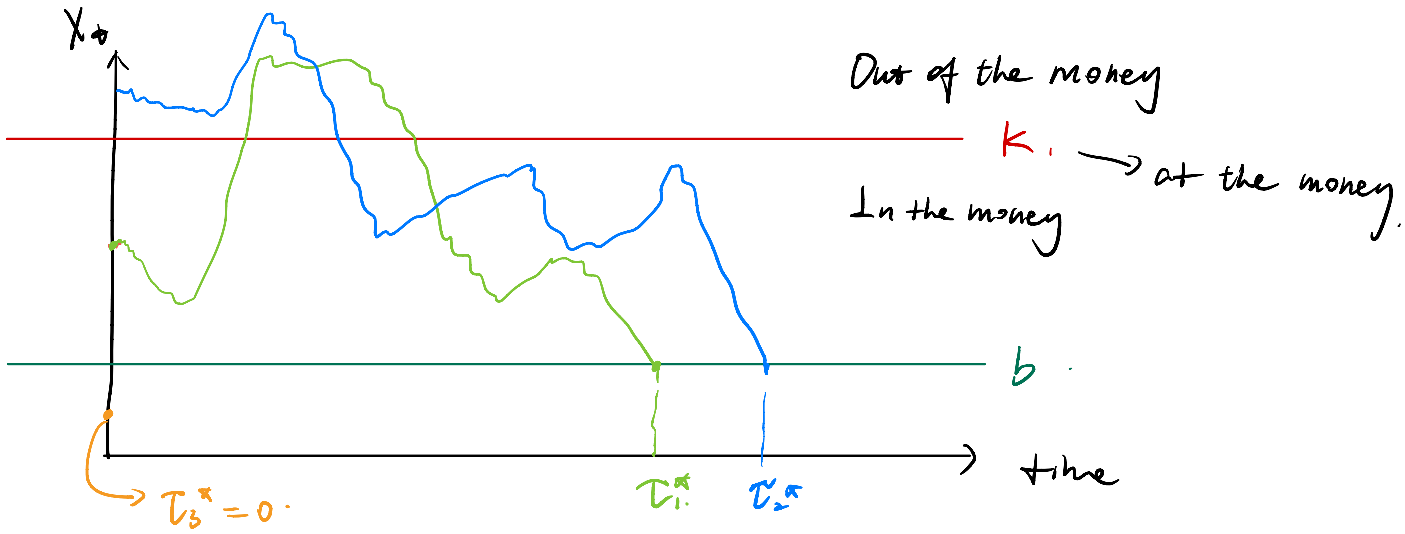
when $x < K$. $g(x) > 0$. but can't wait forever. because of the penalisation in time.

There must be a threshold $b \in (0, K)$, s.t.

$$C = (b, \infty), \quad D = (0, b].$$

- What's the optimal strategy τ^*?

$$\tau^* = \inf\{t \geq 0 : X_t \in D\}.$$



Step 2 Write down the free-boundary problem.

Assume candidate $\hat{V}$ solves.

$$(*) \quad \begin{cases} (r-\delta)x\hat{V}_x + \frac{1}{2}\sigma^2x^2\hat{V}_{xx} - r\hat{V} = 0. & x > b \\ \hat{V}(x) = K - x. & x \leq b. \\ \hat{V}_x(b) = -1. \end{cases}$$

$\hookrightarrow x \leq b < K$.

"Smooth fit".

Step 3   Make an ansatz and solve (*).

Ansatz: $\hat{V}(x) = x^{\gamma}$. plug in (*):

$$(r-\delta)\gamma \cancel{x^{\gamma}} + \frac{1}{2}\sigma^2\gamma(\gamma-1)\cancel{x^{\gamma}} - r\cancel{x^{\gamma}} = 0 \quad , x > b$$

$$\Rightarrow \frac{1}{2}\sigma^2\gamma^2 + (r-\delta-\frac{1}{2}\sigma^2)\gamma - r = 0.$$

quadratic eqn, roots with opposite signs!

$$\gamma_{\pm} = \frac{1}{\sigma^2}\left(-(r-\delta-\frac{1}{2}\sigma^2) \pm \sqrt{(r-\delta-\frac{1}{2}\sigma^2)^2+2\sigma^2 r}\right).$$

when $\gamma_- < 0$. (obvious).

$\gamma_+ \geq 1$. (polynomial $(1) = -\delta \leq 0$).

$\hookrightarrow$ $\gamma_+ = 1$ when $\delta = 0$

- The general solution for $\hat{V}$ is

$$\hat{V}(x) = C_1 x^{\gamma_+} + C_2 x^{\gamma_-}.$$

As $x \to \infty$. $\hat{V}(x) \to 0$. $\Rightarrow C_1 = 0$

(why? when current stock price is high, it's very unlikely to drop below k).

$$\Rightarrow \hat{V}(x) = C_2 x^{\gamma_-}.$$

- "Continuous fit" bdr condition.

$$C_2 b^{\gamma_-} = \hat{V}(b) = g(b) = k - b.$$

$$\Rightarrow \quad C_2 = \frac{k-b}{b^{\gamma_-}}$$

- " Smooth fit ". bdr condition

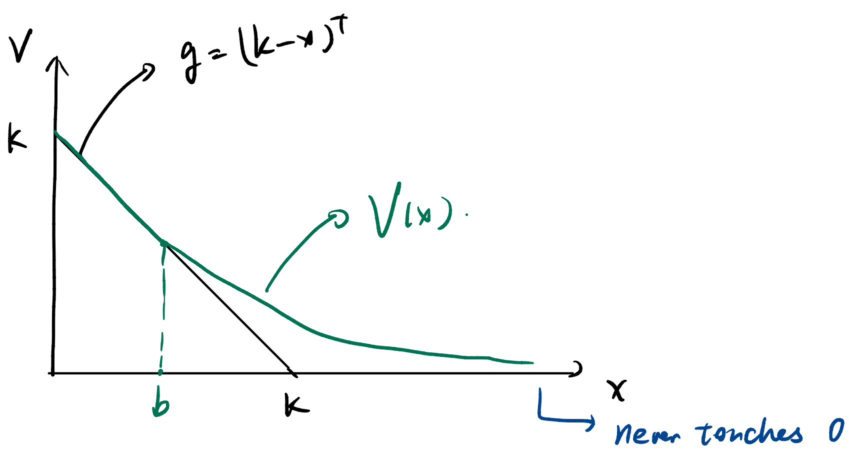
$$(C_2\gamma_-)b^{\gamma_- - 1} = \hat{V}_x(b) = g_x(b) = -1.$$

$$\Rightarrow \quad b = \frac{k\gamma_-}{\gamma_- - 1}$$ (Sanity check: $0 < b < k$).

$$\hat{V}(x) = \begin{cases} k - x, & x \in (0, b]. \\ \dfrac{k-b}{b^{\gamma_-}} x^{\gamma_-}, & x > b. \end{cases}$$

Finally, by the verification thm, $\hat{V} \equiv V$, and τ^* is an optimal strategy.

(How does it look like ?).



What about calls ?

<u>Example 12.2</u> (Perpetual American Call).

Similarly, we consider

$$dX = (r-\delta) X\, dt + \sigma\, dB_t, \qquad X_0 = x.$$

with payoff $g(x) = (x-k)^+$.

Step 1

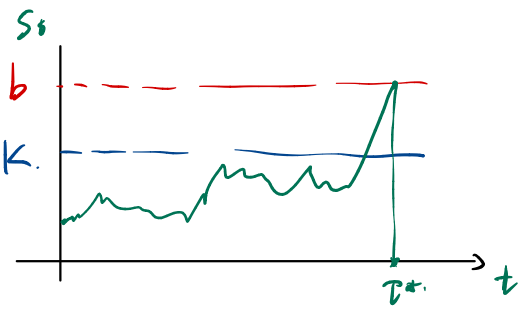
$$V(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-r\tau} (X_\tau - k)^+ \right].$$

There must be a barrier $b > k$. s.t. when X_t is big enough, we exercise. Otherwise we wait.

$C := (0, b)$

$D := [b, \infty)$.

$\tau^* := \inf\{ t : X_t \in D \}$.



Step 2 Let $\hat{V}$ solve:

$$\begin{cases} (r-\delta) x \hat{V}_x + \frac{1}{2}\sigma^2 x^2 \hat{V}_{xx} - r\hat{V} = 0, & x < b \\ \hat{V}(x) = x - k, & x \geq b \\ \hat{V}_x(b) = 1 \end{cases}$$

Step 3 Similarly, we obtain a general solution for $\hat{V}$:

$$\hat{V}(x) = C_1 x^{\gamma_+} + C_2 x^{\gamma_-} \quad \gamma_- < 0.$$

as $x \to 0$, $\hat{V} \to 0$. $\Rightarrow$ only consider γ_+.

plug in "continuous fit" and "smooth fit":

$$C_1 = \frac{b-k}{b^{\gamma_+}}.$$

$$b = \frac{k\gamma_+}{\gamma_+ - 1} \leftarrow \text{ok if } \gamma_+ > 1.$$

i.e. $\delta > 0$.

when $\gamma_+ > 1$.

$$V = \hat{V}(x) = \begin{cases} \frac{b-k}{b^{\gamma_+}} \cdot x^{\gamma_+}; & x \in (0, b). \\ x - k., & x \geq b \end{cases}$$

when $\gamma_+ = 1$: $b = \infty$. it's never optimal to exercise!

Remark. when $T < \infty$, it's still never optimal to exercise an American call early!

pf. Let $\tau \leq T$ be any stopping time. suppose we exercise at τ. we get $(X_\tau - k)^+$, the value at 0 would be

$$\mathbb{E}[e^{-r\tau}(X_\tau - k)^+] \underset{\tau \leq T.}{\leq} \mathbb{E}[(e^{-r\tau}X_\tau - ke^{-rT})_+]$$

Recall. $M_t := e^{-rt}X_t$ is a $\mathbb{Q}$-m.g. for any convex fcn φ:

$$\begin{aligned} \mathbb{E}[\varphi(M_T)] &= \mathbb{E}[\mathbb{E}[\varphi(M_T) | \mathcal{F}_\tau]] \\ &\underset{\text{Jensen's}}{\geq} \mathbb{E}[\varphi(\mathbb{E}[M_T | \mathcal{F}_\tau])] \\ &\underset{\text{m.g.}}{=} \mathbb{E}[\varphi(M_\tau)] \end{aligned}$$

Therefore, since $(x-c)_+$ convex.

$$\mathbb{E}[(e^{-r\tau}X_\tau - ke^{-rT})_+] \le \mathbb{E}[(e^{-rT}X_T - ke^{-rT})_+]$$

$$= \mathbb{E}[e^{-rT}(X_T - k)_+].$$

$\Rightarrow$ Should always wait until T !

Therefore, price of an American call = — " — European call.

(No dividend, $\delta = 0$).

Remark what happens if $T < \infty$?

(put or call when $\delta > 0$).

- Clearly, V is time-dependent. $V = V(t, x)$.
- The exercise threshold becomes time-dependent: $b = b(t)$
- $b(T) = K$. (why? at T we have to make an immediate choice).

The free-boundary now becomes. (e.g. put)

$$\begin{cases} V_t + \mathcal{L}V - rV = 0. & \text{in } C. \\ V(t,x) > (K-x)^+ & \text{in } C \\ V(t,x) = (K-x)^+. & \text{in } D \\ V_x(t,x) = -1. & x = b(t). \end{cases}$$

- Explicit solution is no longer possible. Can study the

Structural properties. .