Lecture 2. Itô integrals.

· why, how ?
· properties ,

(and the associated calculus to manipulate them).

Goal: Define $\int_a^b X(t)\, dW(t)$.

· **Question**: How does the Stieltjes integral get us into trouble?

· Consider two functions. f, g, on $[0,1]$ cont. how do we define the integral of f w.r.t. g ?

· Integration theory in a nutshell : 1) Take simple functions. → piecewise constant

2) Extend the definition by taking limits

$$\int_0^1 f(x)\, dg(x) = \lim_{n\to\infty} \int_0^1 f_n(x)\, dg(x).$$

Is it a good definition ?

Two things to verify: 1) The limit exists

2) The limit is independent of the choice of f_n

We state without proof here :

Lemma 2.1 Suppose g has infinite variation $TV(g,0,1) = \infty$. Then there exists simple functions f_n. which converges to f uniformly, such that $\int_0^1 f_n(x)\, dg(x)$ diverges.

Therefore, 2) fails if we have something with infinite variation!

- Does a BM has infinite variation?

<u>Lemma 2.2</u>. With probability 1, $TV(W, a, b) = \infty$ for any $a<b$.

<u>Remark</u>: i.e. the sample paths of a BM are a.s. of infinite variation, this means a BM can travel infinite distance in a finite time!

(Graph).

Can we weaken the assumptions?

e.g. $X(t) = f(t)$, deterministic.

<u>A new hope:</u> Things look nicer in mean square.

<u>Lemma 2.3</u>. For any partition of the interval $[a, b]$,

$$\sum_i (W_{t_{i+1}} - W_{t_i})^2 \longrightarrow b-a \quad \text{a.s.}$$

"Finite quadratic variation".

<u>Ideas</u> i). Should not allow looking into the future. → adaptedness.

ii). Simply consider a different type of convergence.

↳ L^2.

Def 2.5. (Itô sum).

Let s.p. $X_t \in L^2[a,b]$, i.e.

i) $X_t(\omega)$ is $\mathcal{B}([0,\infty)) \times \mathcal{F}$ - mble.

ii) X_t is $\mathcal{F}_t^W$ - adapted,

iii) $\mathbb{E}[\int_a^b X_t^2 \, dt] < \infty$.

Let P be a partition of $[a,b]$, the Itô sum $I(X,P)$ is defined as follows:

$$I(X,P) := \sum_{j=0}^{n-1} X(t_j) \left(W(t_{j+1}) - W(t_j) \right).$$

Thm 2.6 (Itô integral).

Let P^n denote a sequence of partitions of $[a,b]$, such that $|P_n| := \max_j |t_{j+1} - t_j| \xrightarrow{n\to\infty} 0$, then there exists a r.v. Y with $\mathbb{E}[Y^2] < \infty$, such that

$$\lim_{n\to\infty} \mathbb{E}[(Y - I(X,P^n))^2] = 0.$$

we define the Itô integral $\int_a^b X \, dW = Y$.

Remark: i) This Y does not depend on P^n.

ii). $Y = Y(\omega)$ is a r.v. ((iii) conti. modification?)

Two important properties of Itô integrals

Thm 2.7. (Itô isometry)

Let $X_t \in L^2[a,b]$, then

$$E\left[\left(\int_a^b X_t \, dW_t\right)^2\right] = E\left[\int_a^b (X_t)^2 \, dt\right].$$

finite.

Remark: (on oksendal 3.1.7).

sequence of simple fcn's

$$\begin{tikzcd}
\phi_n(t,w) \arrow[rr, "{L^2([0,\infty)\times\Omega,\ \mu\times\mathbb{P})}"] \arrow[d, "I"'] & & X(t,w) \arrow[d, "I"] \\
I[e_n] \arrow[rr, "{L^2(\Omega,\mathbb{P})}"] & & I[X]
\end{tikzcd}$$

Thm 2.8. (Vanishing expectation).

$$X \in L^2[a,b], \quad E\left[\int_a^b X_t \, dW_t\right] = 0.$$

closely related to m.g.

Ex. 2.3 $\left(\int_a^b W_t \, dW_t\right)$

piecewise constant.

Assume $X_t(w)$ is simple in t. Consider a partition in time,

$$a = t_0 < t_1 < \cdots < t_n = b.$$

Then we can write: $X_t(w) = e_i(w) \cdot \mathbb{1}_{[t_i, t_{i+1}]}$.

$$\int_a^b X_t \, dW_t = \sum_i e_i(w) \cdot (W_{t_{i+1}} - W_{t_i}).$$

If X_t is not simple, e.g. $X_t = W_t$: we approximate it by

$$\sum_i W_{t^*} \cdot \mathbb{1}_{[t_i, t_{i+1}]} \quad \text{for some } t^* \in [t_i, t_{i+1}].$$

Case 1 $t^* = t_i$. (Itô)

$$\int_a^b W_t \, dW_t = \lim_{\sup|\Delta t_i| \to 0} \sum_i B_{t_i} (B_{t_{i+1}} - B_{t_i})$$

$\perp\!\!\!\perp$.

Taking expectation:

$$\mathbb{E}[\text{LHS}] = 0.$$

$\uparrow$

independence.

Case 2 : $t^* = t_{i+1}$. $\rightarrow$ different with Itô?

$$\int_a^b W_t \, dW_t = \lim_{\sup|\Delta t_i| \to 0} \sum_i B_{t_{i+1}} (B_{t_{i+1}} - B_{t_i})$$

$$= \lim_{\sup|\Delta t_i| \to 0} \sum_i (B_{t_{i+1}} - B_{t_i})(B_{t_{i+1}} - B_{t_i} + B_{t_i})$$

$$= \lim_{\sup|\Delta t_i| \to 0} \sum_i \Big(\underbrace{(B_{t_{i+1}} - B_{t_i})^2}_{\mathbb{E} = t_{i+1} - t_i} + \underbrace{B_{t_i}(B_{t_{i+1}} - B_{t_i})}_{\mathbb{E} = 0} \Big)$$

$$\mathbb{E}[\text{LHS}] = \sum_i \Delta t_i = b - a \neq 0.$$

The choice of t^* matters!

Def. 2.9 (Martingale).

An integrable s.p. M_t on $(\Omega, \mathcal{F}, \mathbb{P})$ is a martingale w.r.t. $\mathcal{F}_t$ if

i) $M_t \in \mathcal{F}_t$ for each t.

ii) $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$, for $s \leq t$.

martingale condition.

Remark. A martingale represents a "fair game".

E.X. 2.10. BM is a martingale.

Prf: ii):
$$\mathbb{E}[B_t \mid \mathcal{F}_s] = \mathbb{E}[(B_t - B_s) + B_s \mid \mathcal{F}_s]$$
$$= \mathbb{E}[B_t - B_s \mid \mathcal{F}_s] + B_s$$
$$= B_s. \qquad \square$$

Exercise 1. Check the followings are m.g.'s.

i). $M_t := B_t^2 - t$.

ii) (Doob's m.g.) $M_t := \mathbb{E}[X \mid \mathcal{F}_t]$

Thm 2.11. Itô integral $\int_0^t X \, dW$ is a m.g.

Prf: check the m.g. condition yourselves.

Stochastic Differential Equations.

<u>DE</u>: $dY(t) = K(t)\,dt$.

<u>SDE</u>: $dY(t) = (K(t) + \underbrace{\text{"noise"}}_{}) \,dt$.

- mean 0
- ⊥ increments
- Stationary

Def 2.12 (Itô's process)

An n-dim Itô's process driven by a d-dim BM.

has SDE of the following form:

- $d\underbrace{X_t}_{n\times 1} = \underbrace{\mu_t}_{n\times 1}\,dt + \underbrace{\sigma_t}_{n\times d}\,d\underbrace{W_t}_{d\times 1}$. $\nearrow$ BM.

where μ_t: "drift coeff." σ_t: "diffusion coeff.".

s.p. adapted to $\mathcal{F}_t^W$.

Meaning that:

- $X_t - X(0) = \int_0^t \mu_s\,ds + \int_0^t \sigma_s\,dW_s$.

- A special case: (what we usually use)

Def 2.13 (Itô diffusions).

- X is an Itô diffusion if $\mu = \mu(t, X_t)$. $\sigma = \sigma(t, X_t)$

are deterministic fcn's of t, X_t.

→ why do we care?

- X is an time-homogeneous Itô diffusion if

$$\mu = \mu(X_t), \quad \sigma = \sigma(X_t).$$

⇒ Nice properties w.r.t. stopping times. (later).

- Questions: • Does the SDE have a solution? } today
  - Is it unique?
  - How do we solve it? } next time.

Thm. 2.14. (An existence and uniqueness result)

Let $\mu : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ be mble

$(t,x) \to \mu(t,x)$. $\quad (t,x) \to \sigma(t,x)$

functions satisfying:

$$|\mu(t,x)-\mu(t,y)| + |\sigma(t,x)-\sigma(t,y)| \le C\,|x-y|$$

(Lipschitz).

and

$$|\mu(t,x)| + |\sigma(t,x)| \le D\,(1+|x|).$$ (linear growth).

Then the SDE

$$\begin{cases} dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t \\ X_0 = x_0 \end{cases}$$

↳ deterministic.

has a unique solution $X \in L^2([0,T])$., (Strong solution).

Is there life beyond Lipschitz?

Yes, but we need to be careful:

e.g. Consider ODE's:

i). $dX_t = X_t^2\,dt$ → explode.

$X_t = \frac{1}{1-t}$, $t \in [0, 1)$. , no global solution

ii) $\begin{cases} dX_t = 3X_t^{\frac{2}{3}}\,dt. \\ X_0 = 0 \end{cases}$ → unique

more than one solution.

$X(t) = (t-a)^3 \vee 0$; for $a > 0$.