# Lecture 3. Itô's formula

— with 3 tables.

- Goal: The "chain rule" for stochastic calculus.

- Fundamental calculus: we apply.

$$\int f(x)\,dx = \int \frac{d}{dx}F(x)\,dx = F(x).$$

- Does it work with Itô? $df(B_t) \overset{?}{=} \left( \frac{dB_t}{dt} \cdot f'(B_t) \right) dt.$ doesn't make sense!

- But! BMs has rough paths, nowhere differentiable!

- Instead we look at smt that at least makes sense:

$$df = f'(B_t)\,dB_t + \frac{1}{2} f''(dB_t)^2 + \frac{1}{6} f^{(3)} \cdot (dB_t)^3 + \cdots$$

How many terms do we need?

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Difficult point: BM. If we solve the problem for a BM, we can solve it for many other processes.

( Step 1. BM. )

Thm 3.1 (Itô's formula for BM).

Let $f \in C^2(\mathbb{R})$ (cont. twice differentiable) and $B_t$ be a standard BM

For any $t > 0$.

$$df(B_t) = f'(B_t)\,dB_t + \frac{1}{2} f'' B(t)\,dt.$$

an SDE.

Or in integral form,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,dt.$$

s.p.

Prf of thm 3.1 w.l.o.g. assume $f, f', f''$ bdd.

( On top of this we can approximate any fcn and pass to the limit )

Take a partition with N equal intervals:

$$f(B_t)-f(0) \overset{(*)}{=} \sum_{k=1}^{N} \big( \underbrace{f(B_{t_k}) - f(B_{t_{k-1}})}_{\text{expand.}} \big)$$

By Taylor,

$$f(B_{t_k})-f(B_{t_{k-1}}) = f'(B_{t_{k-1}})(B_{t_k}-B_{t_{k-1}}) + \frac{1}{2} f''(B_{t_{k-1}})(B_{t_k}-B_{t_{k-1}})^2.$$

Then we can write $(*)$ as

$$f(B_t)-f(0) = \sum_{k=1}^{N} f'(B_{t_{k-1}})(B_{t_k}-B_{t_{k-1}}) + \sum_{k=1}^{N} \frac{1}{2} f''(B_{t_{k-1}})(B_{t_k}-B_{t_{k-1}})^2.$$

By the construction of Itô's integral,

$$\boxed{\sum_{k=1}^{N} f'(B_{t_{k-1}})(B_{t_k}-B_{t_{k-1}}) \xrightarrow{L^2} \int_0^t f'(B_s)\,dB_s.}$$

WTS:

$$\sum_{k=1}^{N} f''(B_{t_{k-1}})(B_{t_k}-B_{t_{k-1}})^2 \xrightarrow{L^2} \int_0^t f''(B_{s})\,ds.$$

$$\Longleftrightarrow \boxed{\sum_{k=1}^{N} f''(B_{t_{k-1}})\left[(B_{t_k}-B_{t_{k-1}})^2 - (t_k - t_{k-1})\right] \xrightarrow{L^2} 0}$$

a.s. continuous trajectories.

Idea: $\boxed{\text{"}(dB_t)^2 = dt\text{"}}$

$$\int_0^T (dB_t)^2 = \lim_{N\to\infty} \sum_{k=1}^{N} (\Delta B_{t_{k-1}})^2 \overset{L^2}{=} T = \int_0^T dt$$

$$\mathbb{E}\left[\sum_{k=1}^{N}(\Delta B_{t_{k-1}})^2\right] \underset{\text{ind.}}{=} \sum_{k=1}^{N}\mathbb{E}[(\Delta B_{t_{k-1}})^2] = \sum_{k=1}^{N}\Delta t_{k-1} = T.$$

$$\mathbb{E}\left[\left(\sum_{k=1}^{N}(\Delta B_{t_{k-1}})^2 - T\right)^2\right] = \mathrm{Var}\left(\sum_{k=1}^{N}(\Delta B_{t_{k-1}})^2 - T\right)$$

$$= \sum_{k=1}^{N}\mathrm{Var}\left[(\Delta B_{t_{k-1}})^2\right]$$

$$= \sum_{k=1}^{N}\underbrace{\mathbb{E}[(\Delta B_{t_{k-1}})^4]}_{3(\Delta t)^2} - \underbrace{\mathbb{E}^2[(\Delta B_{t_{k-1}})^2]}_{(\Delta t)^2}$$

$$= 2\left(\frac{T}{N}\right)^2 \cdot N = \frac{2T^2}{N} \longrightarrow 0. \qquad \square$$

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**Corol 3.2**: $f(t,x) \in C^{1,2}([0,\infty)\times\mathbb{R})$. Let $Y_t = f(t, B_t)$, then

$$dY_t = \left(f_t + \frac{1}{2}f_{xx}\right)dt + f_x\, dB_t$$

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**Remark**: $(dB_t)^2$ is computed according to the following table:

| | $dt$ | $dB_t$ |
|---|---|---|
| $dt$ | 0 | 0 |
| $dB_t$ | 0 | $dt$. |

(Drop terms smaller than $dt$.)

Why can we do this? Well, let $p>0$.

$$\int_0^T (dt)^p = \lim_{N\to\infty} \sum_{k=1}^{N} (\Delta t)^p$$

$$= \lim_{N\to\infty} N\cdot\left(\frac{T}{N}\right)^p$$

$$\xrightarrow{N\to\infty} 0 \quad \text{if } p>1.$$

- Choose ansatz.
- Apply Itô
- Integrate

Some examples w.r.t. BM: Solve Itô integrals

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E.X. 3.2 $\quad I = \int_0^t B_s\, dB_s$ (Exercise 2.2)

Let $f(t,x) = \frac{1}{2}x^2$. (why? from calculus).

$Y_t := f(t, B_t) = \frac{1}{2}B_t^2$, $\quad Y_0 = 0$

By Itô:

$$dY_t = B_t\, dB_t + \frac{1}{2}dt$$

$$\frac{1}{2}B_t^2 = Y_t = Y_0 + \int_0^t B_s\, dB_s + \int_0^t \frac{1}{2}\, ds$$

$$= \int_0^t B_s\, dB_s + \frac{1}{2}t$$

$$\int_0^t B_s\, dB_s = \frac{1}{2}(B_t^2 - t).$$

E.x. 3.3 $\quad I = \int_0^t s\, dB_s.$ (Exercise 2.4)

Let $f(t,x) = t x.$ (why?)

$Y_t := t B_t \quad , Y_0 = 0$

$dY_t = B_t\, dt + t\, dB_t + 0 \cdot (dB_t)^2$

$t B_t = \int_0^t B_s\, ds + \int_0^t s\, dB_s$

$\Rightarrow \int_0^t s\, dB_s = t B_t - \int_0^t B_s\, ds$ what's this guy? r.v.

matches integration by parts.

Remark. $\int_0^t s\, dB_s \sim N(0, \frac{1}{3} t^3).$ (by exercise.)

$\int_0^t B_s\, ds$. is also Gaussian.

$$E\left[\int_0^t B_s\, ds\right] = \int_0^t E[B_s]\, ds = 0$$

$$\begin{aligned} Var\left[\int_0^t B_s\, ds\right] &= E\left[\left(\int_0^t B_s\, ds\right)^2\right] \\ &= E\left[\int_0^t \int_0^t B_s \cdot B_u\, ds\, du\right]. \\ &= \int_0^t \int_0^t E[B_s B_u]\, ds\, du \\ &= \int_0^t \int_0^t \min(s,u)\, ds \cdot du. \\ &= \int_0^t \left(\int_0^u s\, ds + \int_u^t u\, ds\right) du. \end{aligned}$$

$$= \frac{1}{3} t^3.$$

$$\int_0^t B_s\, ds \sim N\left(0, \frac{1}{3} t^3\right)$$

**Exercise 1** $\int_0^t B_s^2\, dB_s$.

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(Step 2. 1d Itô process).

· **Thm 3.4** Let $X_t$ be an Itô process with

$$dX_t = \mu_t\, dt + \sigma_t\, dB_t,$$

Let $f(t,x) \in C^{1,2}([0,\infty), \mathbb{R})$, Define $Y_t = f(t, X_t)$.

Then.

$$dY_t = f_t(t, X_t)\, dt + f_x(t, X_t)\, dX_t + \frac{1}{2} f_{xx}(t, X_t)(dX_t)^2$$

$$= f_t\, dt + f_x(\mu_t\, dt + \sigma_t\, dB_t) + \frac{1}{2} f_{xx} (\mu_t\, dt + \sigma_t\, dB_t)^2$$

plug in $dX_t$

$$\boxed{= \left(f_t + f_x \mu_t + \frac{1}{2} f_{xx} \sigma_t^2\right) dt + f_x \sigma_t\, dB_t.}$$

## E.X. 3.5 ( Financial models ).

i) $dX = \mu\, dt + \sigma\, dB_t$. ( Bachelier, 1900 )

"normal model". e.g. In 2020, oil price.

ii) $dX = \mu X\, dt + \sigma X\, dB_t$. ( Geometric BM. Black-Scholes-Merton, 1973)

"lognormal". modern approach.

iii) $dX = \underset{<0}{\beta} X\, dt + \sigma\, dB_t$. (Ornstein-Uhlenbeck ).

"mean reversion". e.g. Vasicek model.

Solution to ii) with Itô:

Let $Y_t = \log(X_t)$.

$$dY_t = \frac{1}{X_t}\, dX_t + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)(dX_t)^2$$

$$= \mu\, dt + \sigma\, dB_t + \left(-\frac{1}{2}\sigma^2\, dt\right)$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dB_t$$

$$Y_t = Y_0 + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma\, dB_s$$

$$= \log(X_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t$$

$$X_t = \exp(Y_t) = X_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\}.$$

Exercise: solve iii)

Ex. 3.6. Calculate $\mathbb{E}[e^{\alpha W_t}]$, $\alpha \in \mathbb{R}$.

- By prob., $X \sim \text{lognormal}(\mu, \sigma^2)$, $\mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2}$.

$$\Rightarrow \mathbb{E}[e^{\alpha W_t}] = \exp(0 + \tfrac{1}{2}\alpha^2 t) = e^{\frac{1}{2}\alpha^2 t}.$$

- By Itô, Let $Y = e^{\alpha W}$, $Y_0 = 1$

$$dY = \alpha e^{\alpha W} dW + \tfrac{1}{2}\alpha^2 e^{\alpha W} dt$$

$$= \alpha Y dW + \tfrac{1}{2}\alpha^2 Y dt$$

$$Y_t = Y_0 + \tfrac{1}{2}\alpha^2 \int_0^t Y_s \, ds + \alpha \int_0^t Y_s \, dW_s$$

$$M_t := \mathbb{E}[Y_t] = Y_0 + \tfrac{1}{2}\alpha^2 \int_0^t \mathbb{E}[Y_s] \, ds + 0$$

$$dM_t = \tfrac{1}{2}\alpha^2 M_t, \quad M_0 = Y_0 = 1$$

$$M_t = M_0 \exp(\tfrac{1}{2}\alpha^2 t) = \exp(\tfrac{1}{2}\alpha^2 t).$$

Exercise: Calculate $\mathbb{E}[X_t]$ where $X_t$ is a GBM. $\left(\mathbb{E}[X_t] = X_0 e^{\mu t}\right)$.

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(Step 3, n-dim Itô).

Recall: $A = [a_{ij}]$, $B = [b_{ij}]$, $m \times n$ matrices. We define their

Frobenius product:

$$\langle A, B \rangle = \sum_{ij} a_{ij} b_{ij} = Tr(AB^T)$$

**Thm 3.7** Let $f(t,x) \in C^{1,2}$, we write

$$f_x = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}, \qquad f_{xx} = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]$$

Let $X_t$ be an n-dim Itô process, i.e.

$$\underset{n\times 1}{dX} = \underset{n\times 1}{\mu}\, dt + \underset{n\times d}{\sigma}\, \underset{d\times 1}{dB_t}$$

$d\times 1$, ind!

Let $Y_t = f(t, X_t)$,

$$dY_t = f_t + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i\, dx_j$$

$$= \underbrace{\left( f_t + \langle \mu, f_x \rangle + \frac{1}{2} \langle \sigma\sigma^T, f_{xx} \rangle \right) dt + \langle f_x, \sigma dB \rangle}_{\text{at } (t, X_t)}.$$

Check!

In other words:

| | $dt$ | $dB_i$ | $dB_j$ |
|---|---|---|---|
| $dt$ | 0 | 0 | 0 |
| $dB_i$ | 0 | $dt$ | 0 |
| $dB_j$ | 0 | 0 | $dt$ |

**Ex. 3.8** (Important).

Denote the Euclidean norm by $\|\cdot\|$, and let $B_t$ be an n-dim BM starting from $x_0 \in \mathbb{R}^n$. Let $f(x) = \|x\|^2$.

$f(B_t) = B_1^2(t) + \cdots + B_n^2(t)$.

$f_x(B_t) = 2B_t$, $f_{xx} = 2I_n$

$$df(B_t) = \left(0 + 0 + \frac{1}{2}\langle I_n, 2I_n\rangle\right)dt + \langle 2B_t, dB_t\rangle.$$

$$= n\,dt + \langle 2B_t, dB_t\rangle.$$

Or, in integral form,

$$\|B_t\|^2 = \|X_0\|^2 + nt + 2\sum_{i=1}^{n}\int_0^t B_i(s)\,dB_i(s).$$

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(Step 4. n-dim correlated Itô).

$\rightarrow$ BM's are correlated.

Suppose the components of BMs are not independent, such that for $s<t$,

$$\operatorname{Corr}(B_i(t)-B_i(s),\, B_j(t)-B_j(s)) = \frac{1}{t-s}\mathbb{E}[B_i(t)-B_i(s),\, B_j(t)-B_j(s)]$$

$$= \rho_{ij}$$

Denote $\rho = [\rho_{ij}]$ as the correlation matrix.

**Thm 3.9.** Itô's lemma still holds, but with $\sigma\sigma^T$ changed to $\sigma\rho\sigma^T$.

i.e. the following table:

| | $dt$ | $dB_i$ | $dB_j$ |
|---|---|---|---|
| $dt$ | $0$ | $0$ | $0$ |
| $dB_i$ | $0$ | $dt$ | $\rho_{ij}\,dt$ |
| $dB_j$ | $0$ | $\rho_{ij}\,dt$ | $dt$. |

## Application

**Thm 3.10** (Martingale representation thm).

Let $M_t$ be an $\mathcal{F}_t^B$-martingale such that $M_T \in L^2$. Then there exists a unique $\mathcal{F}_t^B$-adapted process $\{H_t\}_{t\le T} \in L^2$

$$M_t = M_0 + \int_0^t H_s \, dB_s \quad \text{a.s.} \quad \text{for all } t.$$

↳ deterministic.

**Remark.** We know "an Ito integral is a m.g." This tells us the converse: "Given a m.g. it can be written as an Itô integral."

Why is this important? → application to finance.

$$\underbrace{\Phi_t}_{\text{contract value}} = \Phi_0 + \int_0^t \underbrace{H_s}_{\text{hedging strategy}} \, dB_s.$$

↳ stock price.

$H_t$ is the amount of stock at the beginning of each infinitesimal period.

How to find $H$? e.g. Itô!

In the simple case where $M_t = f(t, B_t)$, by Itô,

$$M_t = M_0 + \int_0^t \left(f_s + \tfrac{1}{2} f_{xx}\right)(s, B_s)\, ds + \int_0^t f_x(s, B_s)\, dB_s.$$

$J_t$ has to be that $\boxed{f_s + \frac{1}{2} f_{xx} = 0.}$ → H.E.

and $\underbrace{H_t(w)}_{\text{process}} = f_x(t, B_t)(w).$

↳ derivative.

e.g. $M_t := B_t^2 - t.$

$$= \int_0^t 2B_s \, dB_s.$$

m.g. rep.

$M_t := \exp\left(\lambda B_t - \frac{1}{2}\lambda^2 t\right).$ → what's this?

(Wald m.g. / GBM)

$$= \lambda \int_0^t e^{\lambda B_s - \frac{1}{2}\lambda^2 s} \, dB_s.$$

m.g. rep.

$$= \int_0^t \lambda M_s \, dB_s.$$

**Exercise.** Find $f(x)$ s.t.

$$M_t = B_t^3 + \int_0^t f(B_s) \, ds. \quad \text{is a m.g.}$$