Lecture 4. PDEs and SDEs : the connection.

- Recall the solution of $dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t$ is called an "Itô diffusion", $\Rightarrow$ diffusion of a dust particle in water.
- Similarly, in the world of PDE's. "diffusion eqns" model the same thing.

For now let's consider the aforementioned Itô diffusions.

Def 4.1. (Infinitesimal generators).

Let X_t be a (time-hom) Itô diffusion $\in \mathbb{R}^n$, for suitable $f: \mathbb{R}^n \to \mathbb{R}$,

The infinitesimal generator $\mathcal{L}$ of X_t is defined as.

$$(\mathcal{L}f)(x) = \lim_{t \downarrow 0} \frac{1}{t}\Big(\mathbb{E}_x[f(X_t)] - f(x)\Big).$$

when the limit exists.

Remark i). notation : "$\mathbb{E}_x$" $-$ $\mathbb{E}[\quad | X_0 = x]$. , "$\mathbb{E}_{t,x}$" $-$ $\mathbb{E}[\quad | X_t = x]$

ii). Why time-homogeneousity?

In the inhomogeneous case, we can make it homogeneous by setting

$dX = 1\,dt + 0\,dB$.

iii). What are suitable f's? e.g, $f \in C_0^2(\mathbb{R}^n)$

$\hookrightarrow$ twice differentiable, compact support.

iv). Set $f(x) = \mathbb{1}_{\{x \in D\}}$. we get infinitesimal change of the prob. distribution.

<u>Thm 4.2</u> Let X_t be 1-dim Itô diffusion, $dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t$

Let $f \in C_0^2(\mathbb{R}^n)$. then $\mathcal{L}f(x)$ exists for all $x \in \mathbb{R}$, and

$$(\mathcal{L}f)(x) = \mu(x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2}.$$

If $X_t \in \mathbb{R}^n$, then

$$(\mathcal{L}f)(x) = \sum_i \mu_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

<u>Prf</u>: (also as an example).

It suffices to show that for a 1-d BM,

$$\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2}.$$

(i.e. the generator of BM is $\frac{1}{2}\Delta$) → Laplacian op.

Apply Itô's formula to $f(B_{t+s})$ with $B_t = x$:

$$\mathcal{L}f(x) = \lim_{s\downarrow 0}\frac{1}{s}\left[E_x[f(B_{t+s})] - f(x)\right]$$

$$= \lim_{s\downarrow 0}\frac{1}{s}\left[E_x\left[\cancel{f(x)} + \underbrace{\int_t^{t+s} f'(B_u)\,dB_u}_{0} + \int_t^{t+s}\frac{1}{2}f''(B_u)\,du\right] - \cancel{f(x)}\right]$$

$$= \frac{1}{2}\frac{d}{ds}\left(\int_t^{t+s} E[f''(B_u)]\,du\right)$$

$$= \frac{1}{2}f''(B_t) = \frac{1}{2}f''(x). \qquad \square$$

<u>Notation</u>: $f = f(t,x) \in C^{1,2}$, still define $(\mathcal{L}f)(t,x) = \mu f_x + \frac{1}{2}\sigma^2 f_{xx}$

$\mathcal{L}$ only applies to x.

E.X. 4.3 (GBM).

X_t solves $dX = rX\,dt + \sigma X\,dB$.

$$(\mathcal{L}f)(x) = rx f_x + \frac{1}{2}\sigma^2 x^2 f_{xx}.$$

What does this generator do ?

Describes the movement of the process in a very small time interval.

- Itô's formula can be written as

$$df(t, X_t) = \frac{\partial f}{\partial t}\,dt + \mu \frac{\partial f}{\partial x}\,dt + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}\,dt + \sigma \frac{\partial f}{\partial x}\,dB_t$$

$$= \Big(\underbrace{\frac{\partial f}{\partial t} + (\mathcal{L}f)(t, X_t)} \Big)\,dt + \sigma \frac{\partial f}{\partial x}\,dB_t.$$

maps to the "drift coefficient".

" $df = (f_t + \mathcal{L}f)\,dt +$ something $\cdot\, dB_t$. "

Stochastic representation for PDE.

- Consider the following PDE of $u(t,x)$

$$\begin{cases} u_t + \mathcal{L}u = 0 \\ u(T, x) = \Phi(x) \end{cases}$$

where $(\mathcal{L}u)(t,x) = \mu u_x + \frac{1}{2}\sigma^2 u_{xx}$.

(Assume this PDE has a $C^{1,2}$ solution).

Apply Itô's formula to $u(t, X_t)$, in the integral form from t to T:

$$\underbrace{u(T, X_T)}_{\Phi(X_T)} = u(t, X_t) + \int_t^T \underbrace{(u_t + \mathcal{L}u)}_{=0,\text{ by PDE}} ds + \underbrace{\int_t^T \sigma u_x \, dB_s}_{\text{things will be nicer if this is } 0.}$$

obs:

Take expectation! Given $X_t = x$, we have

$$\mathbb{E}_{t,x}[\Phi(X_T)] = \underbrace{\mathbb{E}_{t,x}[u(t, X_t)]}_{u(t,x).} + \underbrace{\mathbb{E}_{t,x}\left[\int_t^T \sim dB_s\right]}_{0}$$

$\Rightarrow$ The solution $\boxed{u(t,x) = \mathbb{E}_{t,x}[\Phi(X_T)]}$

solution to a PDE. Expectation of a r.v.

Thm 4.4. (Feynman-Kac formula).

Consider S.P. X_t which solves

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dB_s, \quad \text{with } X_t = x,$$

time horizon / expiring time

Let $D \subset \mathbb{R}^n$ be a connected open domain, $T > 0$. Consider deterministic functions:

$r : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ (discount rate function)

$\Psi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ (running payoff function)

$\Phi : \mathbb{R}^n \to \mathbb{R}$. (final payoff function).

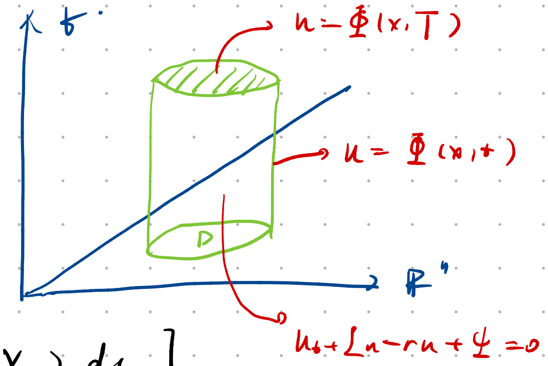
then the unique solution to the PDE:

$$\begin{cases} u_t + \mathcal{L}u - ru + \Psi = 0 & \text{in } [0,T)\times D. \\ u = \Phi & \text{in } ([0,T)\times\partial D)\cup(\{T\}\times D) \end{cases}$$

→ cylinder set

→ Terminal condition

boundary condition



where $u\in C^{1,2}$ is given by

$$u(t,x)=\mathbb{E}_{t,x}\left[d(t,T)\Phi(X_T)+\int_t^T d(t,s)\Psi(s,X_s)\,ds\right].$$

where $d(t_1,t_2)=\exp\left(-\int_{t_1}^{t_2} r(s,X_s)\,ds\right)$

Remark: i) It's easy to derive when $D=\mathbb{R}^n$, $r\geq 0$ constant,

Idea: consider $Y_s=e^{-r(s-t)}u(s,X_s)$ (Exercise).

ii) In finance, Φ: final payoff, Ψ: dividend.

Application: Terminal / Initial value problems.

E.X. 4.5 Solve the PDE
$$\begin{cases} u_t+\frac{1}{2}u_{xx}=0 \\ u(x,T)=x^2 \end{cases}$$

By FK, $u(x,t)=\mathbb{E}_{t,x}[X_T^2]$, where $dX_t=dB_t$.

$$\begin{aligned}
&=\mathbb{E}[B_T^2\mid B_t=x] \\
&=\mathbb{E}[(x+B_{T-t})^2] \\
&=x^2+\mathbb{E}[B_{T-t}^2]+2x\,\mathbb{E}[B_{T-t}] \\
&=x^2+T-t.
\end{aligned}$$

→ Check it in the PDE!

Can we make it stronger?

- Take it to stopping times, restrict it to Itô diffusions.

Thm 4.6 (Dynkin's formula)

Let X_t be an Itô diffusion, and $f \in C_0^2(\mathbb{R}^n)$. Let τ be a $\mathcal{F}_t$-stopping time such that $\mathbb{E}[\tau] < \infty$. then.

$$\mathbb{E}_x[f(X_\tau)] = f(x) + \mathbb{E}_x\left[\int_0^\tau \mathcal{L}f(X_s)\,ds\right].$$

(i.e. The properties we know hold also for stopping times :).

Idea of prf.

i). recall that $\tau \wedge n$ is a stopping time. very commonly used!

ii). (Doob's Optional Sampling thm). "OST".

If X_t is an integrable m.g, and τ is a bounded $\mathcal{F}_t$-stopping time.

then $\mathbb{E}[X_{\tau\wedge t}] = \mathbb{E}[X_0]$.

See the next page for the proof of Dynkin's

Remark. Further reading: "Wald's identities for BM".

[This page is not presented during the lecture].

(proof of Dynkin's formula).

Want: $\mathbb{E}_x\left[\sum_{i,k}\int_0^\tau \sigma_{ik}\underbrace{\frac{\partial f}{\partial x_i}}_{\text{b.d.d.}} dB_k\right] = 0.$

Note that for g b.d.d. $|g| \le M$. then for all integers

$$\mathbb{E}_x\left[\int_0^{\tau\wedge k} g(Y_s)\,dB_s\right] = \mathbb{E}_x\left[\int_0^k \underbrace{g(Y_s)}_{\mathcal{F}_s\text{-mble}}\cdot \underbrace{\mathbb{1}_{s\le\tau}}_{\mathcal{F}_s\text{-mble}}\,dB_s\right] = 0.$$

$$\mathbb{E}_x\left[\left(\int_0^\tau g(Y_s)\,dB_s - \int_0^{\tau\wedge k} g(Y_s)\,dB_s\right)^2\right] \overset{\text{Itô isometry}}{=} \mathbb{E}_x\left[\int_{\tau\wedge k}^\tau g^2(Y_s)\,ds\right].$$

$$\le M^2\cdot\mathbb{E}_x[\tau - \tau\wedge k].$$

Take limit on both sides.

$$\int_0^{\tau\wedge k} g(Y_s)\,dB_s \xrightarrow{L^2} \int_0^\tau g(Y_s)\,dB_s.$$

$$\Rightarrow \int_0^{\tau\wedge k} g(Y_s)\,dB_s \xrightarrow{a.s.} \int_0^\tau g(Y_s)\,dB_s.$$

$$\Rightarrow \mathbb{E}_x\left[\int_0^\tau g(Y_s)\,dB_s\right] = 0.$$

E.X. 4.7 (How long does it take to hit?) → size of domain, distance from the bdr.

Let $B_t \in \mathbb{R}$ be a BM. w. $B_0 = 0$. Let τ be the first time B_t exit from the interval $(-a, a)$?

i) Is $\mathbb{E}[\tau]$ finite? For now let's accept it is. (To prove: F5.)

ii) What is $\mathbb{E}[\tau]$?

By Dynkin's formula, Let $f(x) = x^2$

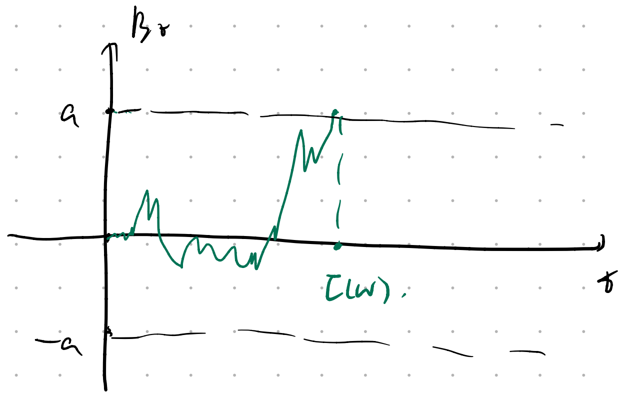
$$\mathbb{E}[f(B_\tau)] = f(B_0) + \mathbb{E}\left[\int_0^\tau \frac{1}{2} f''(B_s)\, ds\right]$$

$$\mathbb{E}[(B_\tau)^2] = 0 + \mathbb{E}\left[\int_0^\tau ds\right] = \mathbb{E}[\tau]$$

$$\Rightarrow \mathbb{E}[\tau] = 1 \cdot a^2 = a^2.$$

This makes sense! $B_t \sim \sqrt{t}$.

(higher dimensions: F5).



Exercise 1. Let $\tau = \inf\{t > 0 : B_t \notin (-a, b)\}$. Determine $\mathbb{E}[\tau]$.

E.X. 4.8 (An interesting example: when Dynkin fails).

Case 1. $\mathbb{E}[\tau] = \infty$.

- Let $T_a := \inf\{t > 0 : B_t = a\}$, $a > 0$, then $\mathbb{E}[T_a] = \infty$.

  Pf. (First step analysis).

Let $\tau_1 = \inf\{t>0 : B_t \in \{-a,a\}\}$.

$\tau_2 = \inf\{t>0 : B_t = 0 \text{ with } B_0 = -a\}$.

Then $\mathbb{E}[\tau_a] = \underset{(0 \to \pm a)}{\mathbb{E}[\tau_1]} + \frac{1}{2}\cdot 0 + \frac{1}{2}\Big(\underset{(-a \to 0)}{\mathbb{E}[\tau_2]} + \underset{(0 \to a)}{\mathbb{E}[\tau_a]}\Big)$

$= \mathbb{E}[\tau_1] + \mathbb{E}[\tau_a]$.

$\mathbb{E}[\tau_a] = \mathbb{E}[\tau_2]$

Only solution: $\mathbb{E}[\tau_a] = \infty$, since $\mathbb{E}[\tau_1]$, $\mathbb{E}[\tau_a] > 0$.

What if we apply Dynkin?

Let $f(x) = x$. by Dynkin

$\mathbb{E}[B_\tau] \overset{?}{=} B_0 + \mathbb{E}\left[\int_0^\tau 0 \, dt\right] = B_0 = 0$.

But $\mathbb{E}[B_\tau] = a$! Dynkin's fails.

<u>Conclusion</u>: $\mathbb{E}[\tau] < \infty$ is an important assumption!

<u>Remark</u>. In this example. $\mathbb{E}[\tau] = \infty$. However $\mathbb{P}(\tau < \infty) = 1$

<u>Case 2</u>. τ is not a stopping time.

"running maximum".

e.g. $\tau = \inf\{t \le 1 : B_t = \max_{t \le 1} B_t\}$.

$\mathbb{E}[B_\tau] = \mathbb{E}[\max_{t\le 1} B_t] > 0$. fails!

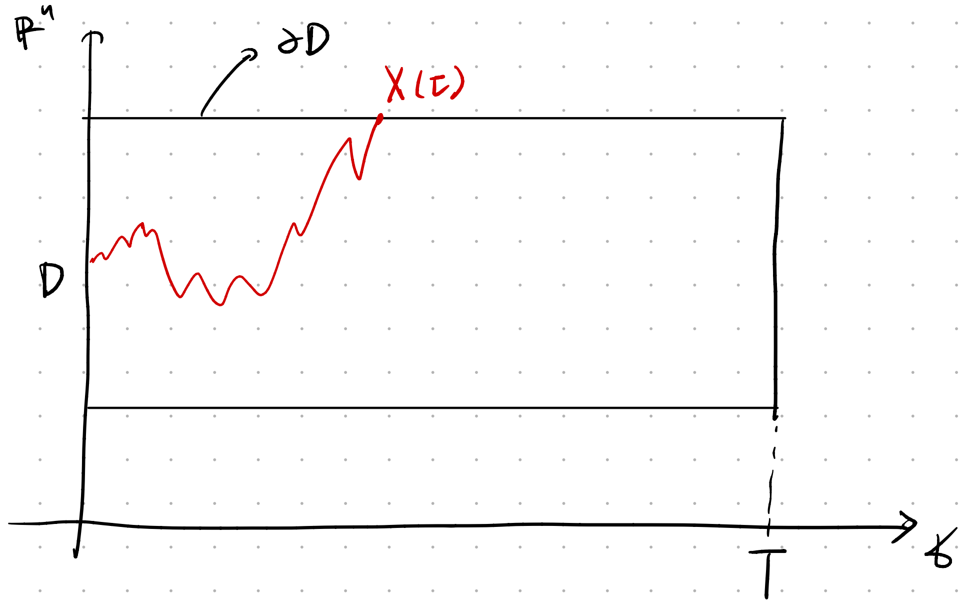
<u>Conclusion</u>: τ is a stopping time is also an important assumption!

• Now take $D \subset \mathbb{R}^n$, $\tau_D := \inf\{t > 0 : X \notin D\}$.

such that $\mathbb{E}_x[\tau_D] < \infty$ for all $x \in D$,

Thm 4.9. Let $\tau_D := \inf\{t : X_t \notin D\}$ and $\tau = \tau_D \wedge T$. Then the Feynman-Kac can be taken up to τ instead of T.

$$u(t,x) = \mathbb{E}_{t,x}\left[d(t,\tau)\, \Phi(X_\tau) + \int_t^\tau d(t,s)\, \Psi(s, X_s)\, ds \right].$$



Next lecture: we take $T \to \infty$.