F6. The heat equation.

Last time: Laplace eqn. $\Delta u = 0$. $u = h(\|x\|)$. (Intro PDE course).

Today: Heat eqn. $u_t = \Delta u$.

Goal: Solve the homogeneous Cauchy IVP

[Any property with Laplace eqn has an (complicated) analogue with HE.]

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Recall. $\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x)$. → both space and time.

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is called the (homogeneous) heat equation. (HE)

Definitions We consider now HE on a bounded domain:

i) Let $D \subset \mathbb{R}^n$ be bounded, open, Let $T \in (0,\infty)$.

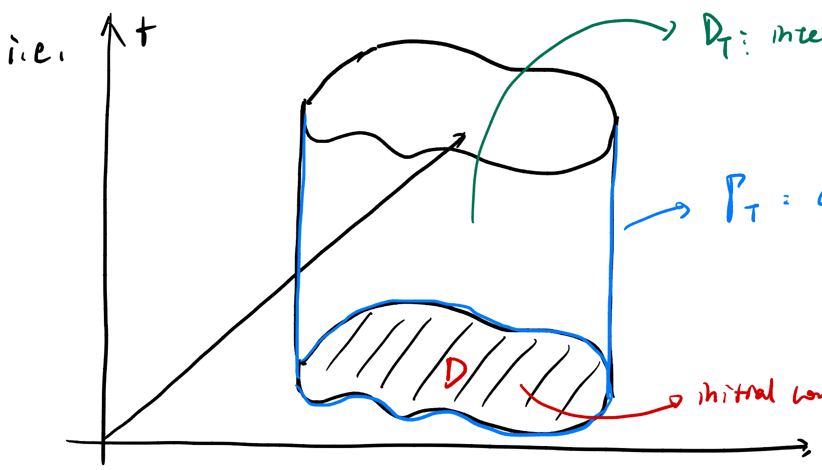
ii) Define the parabolic cylinder with base D as

$$D_T := (0,T] \times D.$$

iii) Define the parabolic boundary of D_T: ↳ "distinguished bdr".

$$\Gamma_T = \overline{D_T} \setminus D_T.$$

i.e.



D_T: interior and the top.

Γ_T: a "cup": bottom and sides

spacial bdr.

initial condition

iii) A boundary value of u at a boundary point is interpreted as the limit : $u(t,x) = \lim_{(s,y)\to(t,x)} u(s,y)$

interior   bdr.

Thm 6.1 (Maximum priciple)

Assume $u \in C^{1,2}(D_T)$ is a solution of the HE in D_T, and extends continuously up to $\overline{D_T}$. Then,

$\to u \in C^{1,2}(D_T) \cap C(\overline{D_T})$.

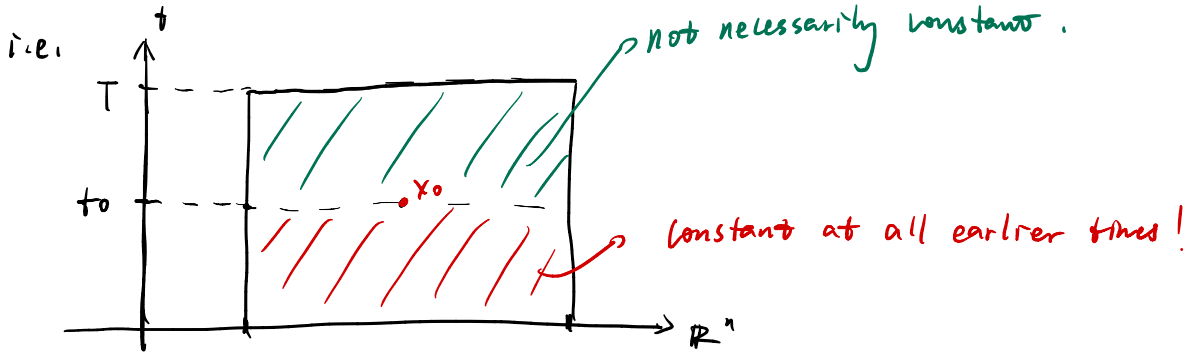
i) $\max_{\overline{D_T}} u = \max_{P_T} u$ (Weak maximum principle).

(global maximum is attained on bdr points)

ii). If D is connected, and there exists a point $(t_0, x_0) \in D_T$. s.t. $u(t_0, x_0) = \max_{\overline{D_T}} u$, then

u is constant in $\overline{D_{t_0}}$.   (Strong maximum principle).

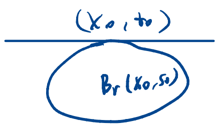
i.e.



Remark . • Similar assertions are valid with "min".

• ii) Intuition: "spikes" will diffuse out over time.

• iii). Argue by extending a ball to the previous times.



• proof see e.g. PDE book by Evans.

Direct consequence : Uniqueness of the solution.

Thm 6.2 (Uniqueness on b.d.d. domains)

Let $\Phi \in C(\Gamma_T)$, $\Psi \in C(D_T)$, assume $u \in C^{1,2}(D_T) \cap C(\bar{D}_T)$ solves

$$(*)\begin{cases} u_t - \Delta u = \Psi & \text{in } D_T. \\ u = \Phi & \text{on } \Gamma_T. \end{cases},$$

then u is unique.

Pf. Assume u_1, u_2 both solve $(*)$, then $\pm(u_1 - u_2)$ both solve

$$\begin{cases} v_t - \Delta v = 0 & \text{in } D_T, \\ v = 0 & \text{on } \Gamma_T. \end{cases}$$

Then $\max(u_1 - u_2) = \min(u_1 - u_2) \overset{\text{wmp}}{=} 0$

i.e. $u_1 - u_2 \equiv 0$. $\square$

What if D is not b.d.d.? Uniqueness still holds for controlled large $|x|$.

Thm 6.3 (Uniqueness for Cauchy IVP). initial value problem.

Let Ψ, Φ be cont, and u solve

$$\begin{cases} u_t - \Delta u = \Psi, & x \in \mathbb{R}^n. \\ u(0,x) = \Phi(x), \end{cases}$$

given that $|u(x,t)| \le A e^{a|x|^2}$, $A, a > 0$.

then u is unique.

Remark. Growth restriction is important. e.g. there are infinitely many solutions to

$$\begin{cases} u_t - \Delta u = 0. & x \in \mathbb{R}. \\ u(0,x) = 0. \end{cases}$$

without the restriction.

each of them grows rapidly except for $u \equiv 0$

Proof. Same as before.

Now we have uniqueness. but do the solutions exist?

Goal: Find a solution to the Cauchy IVP

Recall.
- In F6. we characterised all harmonic fcns of the form $u = h(\|x\|)$, this is called the fundamental solution to the laplace eqn.

  $\Delta u = 0$.

  Intro PDE course.

- In PDE, it's a good strategy to identify some explicit solutions first and further assemble more complicated ones.

What is the fundamental solution of the HE?

Motivation. 1D.
- Let u solve the hom-Cauchy problem ($f=0$)
- $u(ax, a^2t)$ also solves — " —.
- The scale invariance suggests we should consider

$$u(t,x) = v\left(\frac{x^2}{t}\right).$$

- Suppose $u_x \to 0$ as $x \to \pm\infty$, obs.

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} u(t,x)\,dx = 0$$

Conservation of energy.

$$\Rightarrow \quad \int_{\mathbb{R}} u(t,x)\,dx = \text{const}.$$

- However, $\int_{\mathbb{R}} v\left(\frac{x}{\sqrt{t}}\right)dx = \sqrt{t}\int_{\mathbb{R}} v(y)\,dy.$

Thus, scale v by $\frac{1}{\sqrt{t}}$. $\Rightarrow u(x,t) = \frac{1}{\sqrt{t}} v\left(\frac{x}{\sqrt{t}}\right)$!

Thm 6.4. (Fundamental sol. to 1-dim HE). (heat kernel).

Let $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $t>0$, $x \in \mathbb{R}$, then

↳ pdf. $N(0,1)$.

$g(t,x) := \frac{1}{\sqrt{2t}}\varphi\left(\frac{x}{\sqrt{2t}}\right)$ is called then fundamental solution to the 1-dim HE.

Exercise 1 Check $g_t = \Delta g$.

Properties of $g(t,x)$ (check!)

i). If $x \neq 0$, $\lim_{t \downarrow 0} g(t,x) = 0$

ii) If $x = 0$. $\lim_{t \downarrow 0} g(t,x) = \infty$.

(*) Looks familiar ...

iii) $\int_{\mathbb{R}} g(t,x)\,dx = 1$. for all $t>0$ (Exercise).

iv) g is C^∞ in (t,x).

Def 6.5 $\lim_{t \downarrow 0} g(t,x)$ is not a fcn in the usual sense, it is a distribution or generalized fcn called Dirac delta δ:

$$\int_{\mathbb{R}} \delta(x-y) f(y)\,dy = f(x).$$

notional convenience.

Remark: • Imagine a fcn with a centered spike. Or as a measure: unit mass at $x=0$ and 0 elsewhere.

• $\delta(x-y)$ maps test fcns to their value at x.

• δ is the "derivative" of H: (heavyside fcn).

• We can now make sense of the initial data and say $g(t,x)$ solves the Cauchy problem:

$$\begin{cases} g_t = \Delta g \\ \lim_{t\downarrow 0} g(t,x) = \delta(x). \end{cases}$$

Now what? — Use the heat kernel to construct solutions.

Idea: • $g(t,x)$ solves $g_t = \Delta g$, then $g(t,x-y)$ solves it for all x. $x \to (x-y)$ does not change HE.

- $g(t, x-y)\Phi(y)$ solves it as well.

- Linear combination solves it as well: construct

$$\int_{\mathbb{R}} g(t, x-y)\Phi(y)\, dy.$$

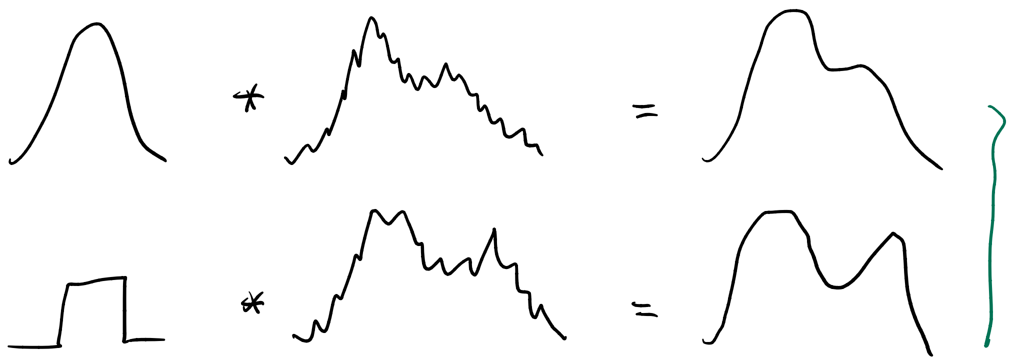
what's this? Convolution!

Recall

- $(f * g)(t) = \int_{-\infty}^{\infty} f(s) g(t-s)\, ds$.

- $f * g = g * f$. (check!)

- Convolution combines smoothness of both fcns.



smoothen.

" Averaging the values of f around t w.r.t. g. "

Thm 6.6 Define $u(t,x) := g(t,x) * \Phi(x) = \int_{-\infty}^{\infty} g(t, x-y)\Phi(y)\,dy$,

then $u(t,x) \in C^\infty((0,\infty)\times\mathbb{R})$, and solves the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 \\ u(0,x) = \Phi(x). \end{cases}$$

for b.d.d and cont. Φ.

Exercise: plug in g and check $(*)$.

Finally: in $\mathbb{R}^n$.

Thm 6.7 (Heat kernel in $\mathbb{R}^n$).

Let $\varphi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{\|x\|^2}{2}\right)$, $x \in \mathbb{R}^n$, be the multivariate standard Gaussian pdf, then

$$g(t,x) = \frac{1}{(\sqrt{2t})^n}\varphi\left(\frac{x}{\sqrt{2t}}\right) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right)$$ is the fundamental solution of HE in $\mathbb{R}^n$.

Remark: i) $\int_{\mathbb{R}^n} g(t,x)\,dx = 1$ (check)

ii) $g(0,x) = \delta(x)$.

- properties and examples : next time.