Lecture 7. Heat eqn. II

Last time: HE. solved Cauchy IVP, fundamental solution

Today. Examples & applications.

Properties of sols to the HE (Exercises § 6)

Consider HE in 1D: $u_t \overset{(*)}{=} u_{xx}$. $0 < t < T$, $a < x < b$.

where $T \in (0, \infty]$, $a \in [-\infty, \infty)$, $b \in (-\infty, \infty]$.

i) Linearity: if u, v solve $(*)$, $\alpha, \beta \in \mathbb{R}$, then $\alpha u + \beta v$ solves $(*)$

ii) Shift & scale: if u solves $(*)$, $\alpha > 0$, $x_0 \in \mathbb{R}$, then $u(\alpha^2 t, \alpha x - x_0)$ solves $(*)$

x_0 α.

for $t \in (0, \alpha^2 T)$, $x \in (\alpha a + x_0, \alpha b + x_0)$.

iii) Differentiation: If $u \in C^3$ and solves $(*)$, then u_t, u_x also solves $(*)$

iv) Integration: if u solves $(*)$ and $v(t,x) := \int_a^x u(t,z)\,dz$, $x \in (a,b)$.

then v solves $(*)$. (Given that $\lim_{z \to a} u_x(t,z) = 0$ for all t).

v). Convolution: if u solves $(*)$. $f : \mathbb{R} \to \mathbb{R}$, then $(u * f)(t,x)$ solves $(*)$

↳ w.r.t. x.

Now we can look at some important examples

(Let I_0 denote the family of fcns that are piecewise cont. and

with at most exponential growth at $\pm\infty$.

Example 7.1.

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R},\ t > 0 \\ u(0,x) = \underbrace{\phi(x)}_{\in I_0} & \end{cases}$$

$u(t,x) = \underbrace{g}_{\text{heat kernel.}} * u_0$: unique solution

Example 7.2 (HE with rate of diffusion k).

$$(*)\quad \begin{cases} u_t - k u_{xx} = 0. & x \in \mathbb{R},\ t > 0,\ k > 0 \\ u(0,x) = \delta_x & \end{cases}$$

Then $g(t,x) = \frac{1}{\sqrt{2kt}} \overset{\text{Gaussian}}{\varphi}\left(\frac{x}{\sqrt{2kt}}\right) = \frac{1}{2\sqrt{\pi k t}} \exp\left(-\frac{x^2}{4kt}\right)$ is the fundamental solution of $(*)$.

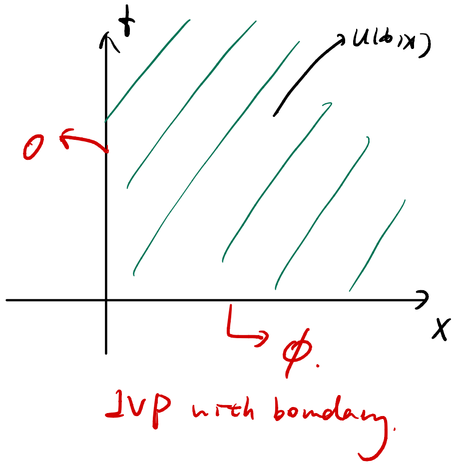
Remark plug in $k = \frac{1}{2}$, Kolmogorov eqn for BM! (later).

Example 7.3 (The Quarter-plane problem). [Tenta 20200604]

Consider the so-called quarter-plane problem:

homogeneous Dirichlet condition

$$\begin{cases} u_t - u_{xx} = 0. & t > 0,\ x > 0. \\ u(t,0) = 0 & t > 0 \\ u(0,x) = \underbrace{\phi}_{\in I_0} & x > 0. \end{cases}$$



Then $v(t,x) = \int_0^\infty \left(g(t,x-y) - g(t,x+y)\right) \phi(y)\, dy$ is a solution.

Def 7.4 A function $f: \mathbb{R} \to \mathbb{R}$ is odd relative to x_0 if

$$f(x_0-y) = -f(x_0+y) \quad . \ \forall\, y\geq 0$$

—— , —— even relative to x_0 if

$$f(x_0-y) = f(x_0+y). \qquad \forall\, y\geq 0.$$

Exercise 1. Show that if f even, h is odd relative to x_0, then $f * h$ is odd relative to x_0.

Prf of 7.3. Idea: extend ϕ to an odd fcn relative to 0, and use 7.1.

Let $\tilde{\phi}$ be the odd extension of ϕ:

$$\tilde{\phi}(x) = \begin{cases} \phi(x), & x>0 \\ -\phi(-x), & x<0 \end{cases}$$

and define $u = \int_{-\infty}^{\infty} g(t, x-y)\, \tilde{\phi}(y)\, dy.$

$$= \int_0^{\infty} g(t,x-y)\,\phi(y)\,dy + \int_{-\infty}^{0} g(t,x-y)\,\phi(y)\,dy$$

$$\text{odd} \rightarrow = \int_0^{\infty} g(t,x-y)\,\phi(y)\,dy - \int_{\infty}^{0} g(t,x+z)\,(-\phi(z))\,dz$$

$$= \int_0^{\infty} (g(t,x-y) - g(t,x+y))\,\phi(y)\,dy.$$

Check:

$$\begin{cases} u_t - u_{xx} = 0. \\ u(0,x) = \tilde{\phi}(x) = \phi(x), \quad x>0. \\ u(t,0) = \int_{\mathbb{R}} \underbrace{g(t,-y)}_{\text{even}}\, \underbrace{\tilde{\phi}(y)}_{\text{odd}}\, dy \underset{\text{odd to 0.}}{=} 0. \end{cases}$$

□

Remark: i) $u(t,x)$ is odd, i.e. the symmetry of $\tilde{\phi}$ is carried to u.

ii). Define $G(t,x,y) \overset{(*)}{=} g(t,x-y) - g(t,x+y)$. G is called the "Green's fcn" of the quater-plane problem.

iii) Important application: Barrier option pricing.

Exercise 2 (Tenta 20210329)

3. (5p) Let $u(t,x)$ be a solution to the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

on $\{(t,x) : t > 0, x > 0\}$ with

$$u(0,x) = u_0(x) \text{ for } x > 0$$

and

$$\frac{\partial u}{\partial x}(t,0) = 0 \text{ for } t > 0.$$

a homogeneous "Neumann condition"

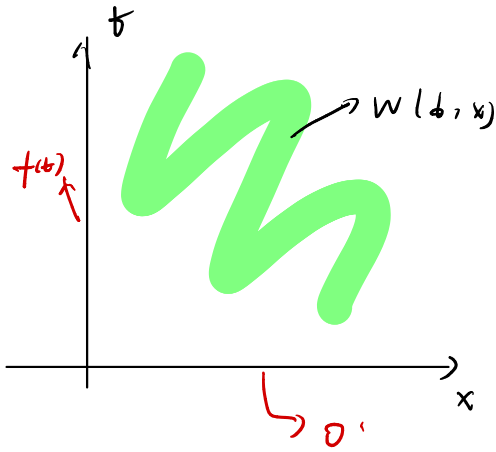
Show that

$$u(t,x) = \int_0^\infty u_0(y) h(t,x,y)\, dy$$

for some function $h(t,x,y)$.

Example 7.5 (Time-varying boundary)

Consider
$$\begin{cases} W_t - W_{xx} = 0 & t,x > 0 \\ W(t,0) = \underline{f(t)} \in I_0 & \\ W(0,x) = 0, & x > 0 \end{cases}$$



Then $w(t,x) = \int_0^t \frac{\partial G}{\partial y}(t-s,x,0) f(s)\, ds$ is a solution.

Green's fcn (*)

Example 7.6 (Initial-boundary value problem for the quarter plane)

$$\begin{cases} V_t - V_{xx} = 0. & t>0, \quad x>0. \\ V(t,0) = f(t). & t>0 \\ V(0,x) = \phi(x). & x>0 \end{cases} \quad \Big\} \; f(0) = \phi(0)$$

Then $V = u + w$, where u solves (7.3), w solves (7.5).

Example 7.7 In the Cauchy IVP, let $\phi(x) \geq 0$. $\phi(x) \not\equiv 0$.

(positive somewhere). then $u(t,x) = \int_{\mathbb{R}} g(t, x-y)\,\phi(y)\,dy > 0$

for all $x \in \mathbb{R}$, $t>0$.

"Infinite propagation speed".

Prf: Strong minimum principle. (Exercise).

Example 7.8 (Recall BS-PDE)

Let S_t be a GBM: $dS_t = r S_t\,dt + \sigma S_t\,dB_t$, where B_t (r: risk-free rate.)

is a BM under $\mathbb{Q}$. If for some financial contract with payoff fcn ϕ,

we define its "price process" Π_t. If for some smooth F,

$$\Pi_t = F(t, S_t)$$

e.g. $\phi(S) = (S-k)^+$ "call option".

holds, then F solves

$$(*)\begin{cases} F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF = 0 \\ F(T,s) = \phi(s). \end{cases}$$

(B S-PDE)

Furthermore, by FK thm.

$$\boxed{F(t,s) = e^{-r(T-t)}\mathbb{E}^{Q}_{t,s}[\phi(S_T)]}$$

(risk-neutral valuation formula)

B-S PDE is equivalent to an IVP of the heat equation.

Apply the change of variable:

$$\begin{cases} \tau = \frac{1}{2}\sigma^2 (T-t) \\ x = \log s. \end{cases}$$

and assume $F(t,s) = v(\tau, x) = v(\frac{1}{2}\sigma^2(T-t), \ln s)$, then

$$F_t = v_\tau \cdot \tau_t = -\frac{1}{2}\sigma^2 v_\tau$$

$$F_x = \frac{1}{s} v_x$$

$$F_{xx} = -\frac{1}{s^2} v_x + \frac{1}{s^2} v_{xx}$$

Want to get rid of.

plug in $(*)$

$$\Rightarrow \begin{cases} v_\tau + (1 - \frac{2r}{\sigma^2}) v_x - v_{xx} + \frac{2r}{\sigma^2} v = 0, \quad (\tau, x) \in [0, \frac{\sigma^2}{2}T] \times \mathbb{R}. \\ v(0, x) = \phi(e^x). \end{cases}$$

Let $u(\tau,x) = e^{-\alpha x - \beta\tau} v(\tau, x)$. for some α, β. then

(exercise). Let $k = \frac{2r}{\sigma^2}$, $\alpha = \frac{1}{2}(1-k)$, $\beta = -\frac{1}{4}(k+1)^2$,

u then solves.

$$\begin{cases} u_\tau - \Delta u = 0, & (\tau, x) \in [0, \frac{\sigma^2}{2}T] \times \mathbb{R}. \\ u(0, x) = e^{-\alpha x} \phi(e^x) & \end{cases}$$

Example 7.9. Pricing barrier options.

Recall. $C(t, s)$ – Pricing fcn of an European call option with strike K and expiration T.

Exercise only at T, $\phi = (S_T - k)^+$

↳ stock at T.

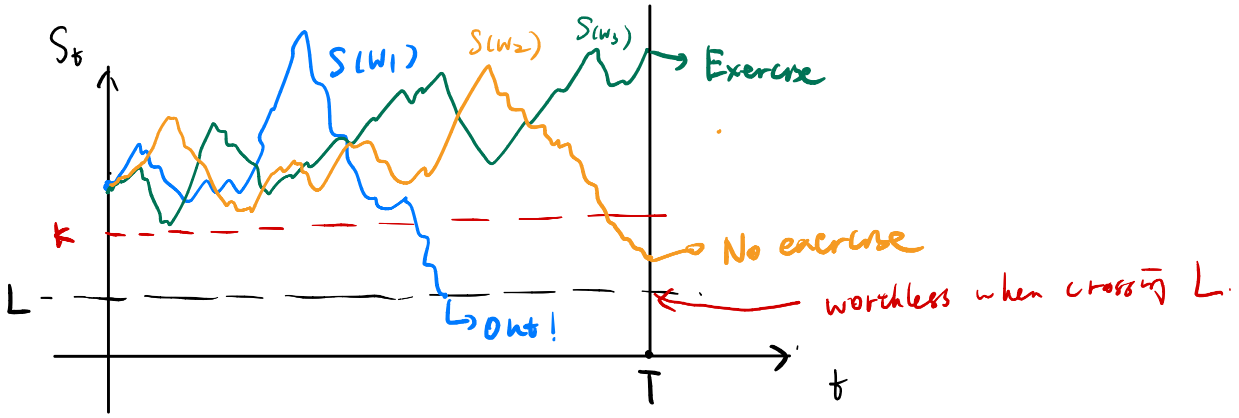
Want: The price of the down-and-out version with barrier $L \in (0, K)$.

"knock-out contract".

↖ compatibility.

- path-dependent:

$$\text{payoff at } T = \begin{cases} \phi(S_T), & \text{if } S_t > L \text{ for all } t \le T \\ 0, & \text{if } S_t \le L \text{ for some } t \le T. \end{cases}$$



- Short-hand notation "DO".

$$\phi(S_T) = (S_T - k)^+ \mathbb{1}_{\{\min_{t \le T} S_t > L\}}.$$

- Other barrier types:
  - "DI": down-and-in. $\phi(S_T) = (S_T - k)^+ \mathbb{1}_{\{\min_{t \le T} S_t \le L\}}.$

- "UO" : up-and-out $\phi(S_T) = (S_T - K)^+ \mathbb{1}_{\{\max_{t \le T} S_t < L\}}$.
- "UI" up-and-in $\phi(S_T) = (S_T - K)^+ \mathbb{1}_{\{\max_{t \le T} S_t \ge L\}}$.

Thm 7.10 For European type options, the followings hold:

$$\text{DO-call} + \text{DI-call} = \text{UO-call} + \text{UI-call} = \text{Call}.$$

$$\text{DO-put} + \text{DI-put} = \text{UO-put} + \text{UI-put} = \text{put}.$$

Pf. Trivial. (Intuition: less chance of getting validated — cheaper).

- With arbitrage pricing, need to know the probability density fcn of the "absorbed process" $S_{t \wedge \tau}$, where $\tau = \inf\{t : S_t = L\}$.

Thm 7.11 The price of a DO-call $F(t,s)$ satisfies

$$F(t,s) = C(t,s) - \left(\frac{s}{L}\right)^{1-\frac{2r}{\sigma^2}} C\left(t, \frac{L^2}{s}\right), \quad s > L.$$

(See Björk Thm 18.8)

Today: solve it using PDE's. "method of image".

Recall.

$$\frac{\text{BS-eqn}}{F(t,s)} \longleftrightarrow \frac{\text{HE}}{u(\tau, x)}$$

$$\tau = \frac{\sigma^2}{2}(T-t), \quad x = \log s.$$

$F(t,s) = e^{\alpha x + \beta \tau} u(\tau, x)$, where $\alpha = \frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)$, $\beta = -\frac{1}{4}\left(1 + \frac{2r}{\sigma^2}\right)^2$.

In our case: Need the correct initial data for DO!

Obs. i) For the regular call:

$$\phi(x) \left(:= u(0,x) \right) = e^{-\alpha x} F(T,S) = e^{-\alpha x}(e^x - K)^+.$$

ii) $S > L \iff x > \log L$

$\Rightarrow$ Solve only for $x > \log L$, with $u(t, \log L) = 0$ all t.

Idea: modify ϕ to make it odd relative to $x = \log L$.

(recall Example 7.3)

iii) looking for $u(\tau, x)$. s.t.

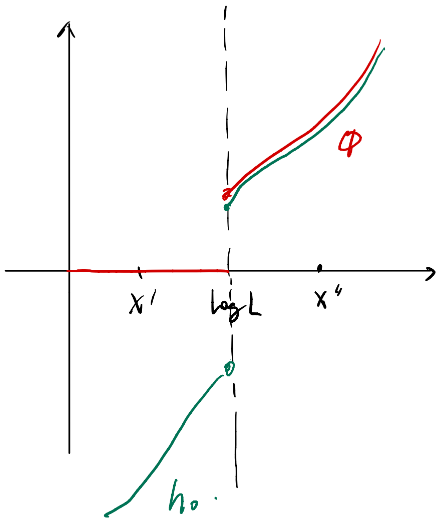
$$u(\tau, 2\log L - x) = -u(\tau, x) \quad \text{for all } x.$$

Step 1. Extend ϕ by odd symmetry to whole space.

we call the extension h_0. then

$$h_0(x) = \phi(x) - \phi(2\log L - x).$$

check!



Step 2. Solve the whole space problem. (convolution)

$g * h_0$ solves $\begin{cases} u_t - \Delta u = 0 \\ u(0,x) = h_0 \end{cases}$

Then, u smooth, u odd w.r.t. $\log L$. $\Rightarrow u(t, \log L) = 0$.

Step 3. Calculate value of the option.

Obs that $(g * \phi)(\tau, x) = e^{-\alpha x - \beta\tau} C(t, e^x)$ regular call.

$$(g * h_0)(\tau, x) = (g * \phi)(\tau, x) - \underbrace{(g * \phi(2\log L - \cdot))(\tau, x)}_{(g*\phi)(\tau,\, 2\log L - x)}$$

linear.

$$= e^{-\alpha(2\log L - x) - \beta\tau} C(t, e^{2\log L - x})$$

Therefore, $F(t,s) = e^{\alpha x + \beta\tau}(g * h_0)(\tau, x)$

$$= C(t,s) - e^{\alpha x - \alpha(2\log L - x)} C(t, e^{2\log L - x})$$

$\log L - x = \log \frac{L}{s}$

$$\boxed{= C(t,s) - \left(\frac{s}{L}\right)^{\left(1 - \frac{2r}{\sigma^2}\right)} C\left(t, \frac{L^2}{s}\right).}$$

where $S > L$. ☒

(See Kohn section 2. P7-9).

Comment: compatibility issue when decomposing.