Lecture 10. Markov processes and Kolmogorov eqns

Today: Distribution of a BM at t?

Def 8.1. Let $X_t \in \mathbb{R}^n$ be a S.P. and $\mathcal{F}_t$ be its natural filtration.

We say X has the Markov property if for any $s \in [0,t]$ and any b.d.d. Borel fcn f:

$\sigma(X_s)$.

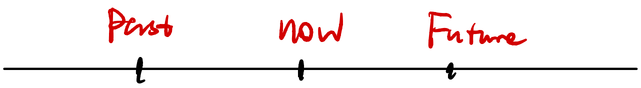
$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s]$$

Equivalently, for any Borel set $A \in \mathbb{R}^n$.

$$\mathbb{P}(X_t \in A \mid \mathcal{F}_s] = \mathbb{P}(X_t \in A \mid X_s).$$

If X has the markov property, we call a Markov process.

Intuition:



The future does not depend on how we got here. "Memoriless"

Remark. If for any stopping time τ, on event $\{\tau < \infty\}$. X satisfies

for each $t \geq 0$. $\mathbb{E}[f(X_{\tau+t}) \mid \mathcal{F}_\tau] = \mathbb{E}[f(X_{\tau+t}) \mid X_\tau]$, we say

X has the Strong Markov property.

(Strong MP $\Rightarrow$ MP). $\to$ take $\tau = t$.

Thm 8.2. i). BM is a (strong) Markov process

ii) The Itô diffusion $dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$

is a (strong) Markov process.

Prf : Oksendal §7.1.2. §7.2.4.

Remark. Independent increments $\Rightarrow$ Markov property

$\not\Leftarrow$ e.g. $X_{n+1} - X_n = N(X_n, 1)$.

Def 8.3 Consider a Markov process $X_t \in \mathbb{R}^n$. we define the transition probability measure at time t, from state x at time $s < t$ by

$$P(A, t; x, s) = \mathbb{P}(X_t \in A \mid X_s = x).$$

where A is a Borel subset of $\mathbb{R}^n$.

Remark. i) In the continuous case, $\mathbb{P}(X_s = x) = 0$. We then interpret

$$P(A, t; x, s) = \mathbb{E}_{s,x}[\mathbb{1}_A(X_t)]$$

ii). Assume the transition measure has a density, we denote it by $p(y, t; x, s)$, the "transition PDF from x at s to y at t":

$$p(y,t;x,s) = f_{X_t|X_s}(y|x)$$ → conditional density.

Obviously, $P(A,t;x,s) = \int_A p(y,t;x,s)\,dy$, and

"Lec A eapand" $1 = \int_{\mathbb{R}^n} p(y,t;x,s)\,dy$.

iii) We call the pair (y,t) "forward variables" and

(x,s) "backward variables".

Thm 8.4 (Chapman-Kolmogorov eqn)

For Markov process X_t, we have

$$p(y,t;x,s) = \int_{\mathbb{R}^n} p(y,t;z,u)\,p(z,u;x,s)\,dz, \quad s \le u \le t.$$

Intuition. Want to get from (x,s) to (y,t), choose some intermediate time u, summing up the probabilities for all possible location z.

Prf. (Not presented in lecture).

(1D) $$p(y,t;x,s) = f_{X_t|X_s}(y|x) \left(= \frac{f_{X_t,X_s}(y,x)}{f_{X_s}(x)} \right)$$

$$= \frac{\int_{\mathbb{R}} f_{X_t,X_u,X_s}(y,z,x)\,dz}{f_{X_s}(x)}$$

$$= \int_{\mathbb{R}} f_{(X_t,X_u)|X_s}((y,z)|x)\,dz.$$

$$= \int_{\mathbb{R}} f_{X_t \mid (X_u, X_s)}(y \mid (z, x))\, f_{X_u \mid X_s}(z \mid x)\, dz.$$

$$= \int_{\mathbb{R}} f_{X_t \mid X_u}(y \mid z)\, f_{X_u \mid X_s}(z \mid x)\, dz \qquad \boxtimes$$

Now assume X_t is an Ito diffusion:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t.$$

Recall. Infinitesimal generator $\mathcal{L}$:

$$(\mathcal{L} g)(x) = \sum_i \mu_i \frac{\partial}{\partial x_i} g + \frac{1}{2} \sum_{j,k} C_{jk} \frac{\partial^2}{\partial x_j \partial x_k} g.$$

where $[C_{jk}] = \sigma \sigma^T$.

Thm 8.5 (Kolmogorov Backward Eqn).

1st form. Satisfied by transition measure P as a function of the backward variables. Fix $A \subset \mathbb{R}^n$, $t > 0$.

$$\boxed{\begin{array}{l} \dfrac{\partial P}{\partial s}(A, t; x, s) + \mathcal{L} P(A, t; x, s) = 0, \quad (s, x) \in (0, t) \times \mathbb{R}^n \\ \qquad\qquad P(A, t; x, t) = \mathbb{1}_A(x). \end{array}}$$

on x.

(pf: F-K thm, $r = 0$. $\psi = 0$. $\Phi = \mathbb{1}_A$).

2nd form. Satisfied by the density p as a function of the backward variables. Fix $y \in \mathbb{R}^n$, $t > 0$.

$$\boxed{\begin{aligned} &\frac{\partial p}{\partial s}(y,t;x,s) + \mathcal{L}\, p(y,t;x,s) = 0, \quad (s,x)\in(0,t)\times\mathbb{R}^n \\ &p(y,t;x,s) = \delta(x-y) \quad \text{as } s\to t. \end{aligned}}$$

on x.

"Let A shrink".

Now we look at the adjoint of $\mathcal{L}$, $\mathcal{L}^*$

$$\mathcal{L}^* g = -\sum_i \frac{\partial}{\partial x_i}(\mu_i g) + \frac{1}{2}\sum_{j,k}\frac{\partial^2}{\partial x_j \partial x_k}(C_{jk}\, g)$$

<u>Remark</u>. For smooth g, h, s.t. things vanish at infinity.

$$\langle \mathcal{L} g, h\rangle_{L^2} = \langle g, \mathcal{L}^* h\rangle_{L^2} \qquad \text{(Exercise 1)}$$

<u>Thm 8.6</u>. (Kolmogorov forward Eqn / Fokker-Planck).

<u>1st Form</u>.

- satisfied by the transition density p w.r.t the forward variables

Fix $s>0$. we have.

$$\boxed{\begin{aligned} &\frac{\partial p}{\partial t}(y,t;x,s) - \mathcal{L}^* p(y,t;x,s) = 0, \quad (t,y)\in(s,\infty)\times\mathbb{R}^n \\ &p(y,t;x,s) = \delta(y-x) \quad \text{as } t\to s. \end{aligned}}$$

on y

2nd form. It describes the probability distribution by solving an IVP

Let $p(t,x)$ be the probability density of X_t, $p_s(x)$ be the initial density, then.

$$\boxed{\begin{aligned} \partial_t p - \mathcal{L}^* p(t,x) &= 0 \\ p(s,x) &= p_s(x) \end{aligned}}$$

Prf. Assume that X_t has a transition prob. density, with $X_s = x$.

(1st form, 1d.). Let $h \in C_0^2(\mathbb{R})$. Apply Itô to h for $t > s$

$$h(X_t) = h(x) + \int_s^t \underbrace{\mu h_x + \tfrac{1}{2}\sigma^2 h_{xx}}_{\mathcal{L}h(u,X_u)} \, du + \int_s^t h_x(u, X_u)\, dB_u.$$

$$\mathbb{E}_{s,x}[h(X_t)] = h(x) + \mathbb{E}_{s,x}\left[\int_s^t (\mathcal{L}h)(X_u)\, du\right]$$

$$\text{RHS} = h(x) + \int_s^t \mathbb{E}_{s,x}[(\mathcal{L}h)(X_u)]\, du$$

$$\| \qquad = h(x) + \int_s^t \int_{\mathbb{R}} (\mathcal{L}h)(y)\, p(y,u;x,s)\, dy\, du.$$

$$\text{LHS} = \int_{\mathbb{R}} h(y)\, p(y,t;x,s)\, dy.$$

Differentiate both w.r.t. t:

$$\int_{\mathbb{R}} h(y) \frac{\partial p}{\partial t}(y,t;x,s)\, dy = \int_{\mathbb{R}} (\mathcal{L}h)(y)\, p(y,t;x,s)\, dy$$

$$= \int_{\mathbb{R}} h(y)\, (\mathcal{L}_y^* p)(y,t;x,s)\, dy.$$

adjoint

The choice of h is arbitrary $\Rightarrow$ $\frac{\partial p}{\partial t} = \mathcal{L}_y^* p$. ☒

Remark. The KFE is weaker than the KBE.

↳ take derivative of μ, σ, but not necessarily exist!

Example 8.7 (Special case: BM).

Let $\rho(t,x)$ be the density of a BM, $u(t,x) = \mathbb{E}[f(X_T)]$

KFE. $\quad \rho_t - \frac{1}{2}\Delta \rho \overset{(+*)}{=} 0, \quad \rho(0,x) = \delta_0. \quad (t > s)$

KBE $\quad u_t + \frac{1}{2}\Delta u \overset{(-*)}{=} 0, \quad u(T,x) = f(x) \quad (t < T)$

Obs. $\mathcal{L} = \mathcal{L}^*$, $(+*)$ is $(-*)$ reversed (in time).

⟶ This is an accident! (Self-adjointness of the Laplacian).

Example 8.8 (BM with drift)

Consider $dX_t = \mu\, dt + dB_t$. with $X_s = 0$. For $t > s$,

$$X_t = \mu(t-s) + (B_t - B_s).$$

Obs. that $p(y,t;0,s) = \frac{\partial}{\partial y} P[X_t \le y \mid X_s = 0]$, where

$$\mathbb{P}(X_t \le y \mid X_s = 0) = \mathbb{P}\left(\frac{B_t - B_s}{\sqrt{t-s}} \le \underbrace{\frac{y - \mu(t-s)}{\sqrt{t-s}}}_{f(y)}\right) = N(f(y))$$

↓ cdf of Gaussian.

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} f(y)^2\right) \cdot \frac{\partial}{\partial y} f(y)$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-\mu(t-s))^2}{2(t-s)}\right)$$

The KFE is:

$$\frac{d}{dt} p(y,t;0,s) + \underbrace{\mu \frac{d}{dy} p(y,t;0,s) - \frac{1}{2}\frac{d}{dy^2} p(y,t;0,s)}_{\mathcal{L} + \mathcal{L}^*} = 0$$

Remark. i) KFE is in general difficult to solve analytically.

ii) An easier problem: large-time behaviour of a diffusion.

- Assume X_t is time-homogeneous, $\mu = \mu(x)$, $\sigma = \sigma(x)$. Is there a probability that doesn't change with time?

Recall. Stationary distribution for a Markov chain:

$$\pi = \pi P.$$

$\hookrightarrow$ transition matrix.

Similarly, we can define it for diffusions.

Def 8.9. (Stationary distribution).

A stationary distribution or invariant measure is a probability distribution μ that doesn't change with time, i.e.

$$\mathbb{P}(X_t \in A \mid X_0 \sim \mu(x)) = \mu(A).$$

for all mble sets $A \in \mathbb{R}^n$. If the process has a density $\rho(t,x)$, then it's a stationary density if

$$\rho_t = 0. \iff \mathcal{L}^* \rho = 0.$$

Application 1 Describe large time statistics (stationary distribution) of a process : $\rho_\infty(x) = \lim_{t \to \infty} \rho(t, x)$. (if exists).

Example 8.10 (OU-process).

Let $dX_t = \overset{<0}{\mu} X_t + \sigma \, dW_t$. KFE gives.

$$\frac{\partial \rho}{\partial t} - \left(- \frac{\partial}{\partial x} (\mu x \rho) + \frac{1}{2} \sigma^2 \frac{\partial^2 \rho}{\partial x^2} \right) = 0. \quad \rho = \rho(t,x).$$

- In steady state: $\mathcal{L}^* \rho_\infty = 0$ yields $-\frac{\partial}{\partial x}(\mu x \rho_\infty) + \frac{1}{2} \frac{d^2}{dx^2} \rho_\infty = 0.$
- Integrating: $\underbrace{-\mu}_{>0} x \rho_\infty + \frac{1}{2} \sigma^2 \frac{\partial \rho_\infty}{\partial x} = C.$ ↳ "probability flux".
- As $|x| \to \infty$. we expect $\rho_\infty \to 0$ faster than $\frac{1}{|x|}$, since $\int_{\mathbb{R}} \rho_\infty = 1.$

This implies $C = 0.$

- $\frac{\partial \rho_\infty}{\partial x} / \rho_\infty = \frac{2 \mu x}{\sigma^2} \implies \ln \rho_\infty = \frac{\mu}{\sigma^2} x^2 + D.$

$$\Rightarrow \quad p_\infty = D \exp\left(\frac{\mu}{\sigma^2} x^2 \right).$$

$\cdot \int_{\mathbb{R}} p_\infty = 1 \xRightarrow{\text{solve } D}$ $\boxed{p_\infty = \frac{1}{\sqrt{2\pi\sigma^2/(-2\mu)}} \exp\left(-\frac{1}{2} \frac{x^2}{\sigma^2/(-2\mu)} \right)}$

Gaussian!

Therefore, the stationary distribution is Gaussian, with mean 0, variance $-\sigma^2/2\mu$.

$\longrightarrow$ Check with Assignment 1!

Remark i) Some processes have no stationary distribution

e.g. BM. (see exercises).

ii) OU is the only (up to a change of variable)

Markovian, stationary, Gaussian process.

$\boxed{\text{Application 2}}$ Determining moments. (Later).