Question 1.

i). $\mu(t,x) = \mu x$. $\sigma(x) = \sigma$ satisfy the Lipschitz and linear growth conditions. Therefore strong solution.

Let $Y_t := e^{-\mu t} X_t$, $Y_0 = x$.

$$dY_t = -\mu Y_t \, dt + e^{-\mu t} dX_t$$

$$= -\mu Y_t \, dt + e^{-\mu t} (\mu X_t \, dt + \sigma \, dW_t)$$

$$= \sigma e^{-\mu t} \, dW_t$$

$$Y_t = x + \sigma \int_0^t e^{-\mu s} \, dW_s$$

$$X_t = e^{\mu t} \cdot Y_t$$

$$= x e^{\mu t} + \sigma \int_0^t e^{\mu(t-s)} \, dW_s.$$

ii) $$\mathbb{E}[X_T] = x e^{\mu T} + \sigma \underbrace{\mathbb{E}[\int_0^T \sim dW_s]}_{0} = x e^{\mu T}.$$

$$\mathbb{E}[X_T^2] = \mathbb{E}[(x e^{\mu t} + \sigma \int_0^t e^{\mu(t-s)} \, dW_s)^2]$$

$$= x^2 e^{2\mu t} + \mathbb{E}[\int_0^t \sigma^2 e^{2\mu(t-s)} \, ds]$$

$$= x^2 e^{2\mu t} + \sigma^2 e^{2\mu t} \cdot \frac{1}{2\mu}(1 - e^{-2\mu t})$$

$$= x^2 e^{2\mu t} + \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)$$

Variance follows.

iii) Directly from Feynman – Kac.

Terminal condition: $u(T,x) = e^{i\xi x}$

iv)
$$\begin{cases} u_t + \mu x u_x + \frac{1}{2}\sigma^2 u_{xx} = 0. \\ u(T,x) = e^{i\xi x} \end{cases}$$

$$u(t,x) = \exp(\beta(t) + i\xi \alpha(t) x).$$

$$\begin{cases} u_t = (\beta_t + i\xi x \alpha_t)\, u. \\ u_x = i\xi\alpha(t) \cdot u. \\ u_{xx} = -\xi^2\alpha^2(t) \cdot u. \end{cases}$$

$$\beta_t + i\xi x \alpha_t + \mu x\, i\xi\alpha(t) - \frac{1}{2}\sigma^2\xi^2\alpha^2(t) = 0$$

$$\Rightarrow i\xi(\alpha_t + \mu\alpha(t))\, x + \beta_t - \frac{1}{2}\sigma^2\xi^2\alpha^2(t) = 0$$

for all $t \leq T$, all x.

$$\Rightarrow \alpha_t + \mu\alpha(t) \overset{(*)}{=} 0 \quad , \quad \beta_t \overset{(**)}{=} \frac{1}{2}\sigma^2\xi^2\alpha^2(t).$$

(*) yields
$$\begin{cases} \alpha(t) = C e^{-\mu t}. \\ \alpha(T) = 1 \end{cases} \Rightarrow \alpha(t) = e^{\mu(T-t)}$$

(**) yields.
$$\begin{cases} \beta'(t) = \frac{1}{2}\sigma^2\xi^2 e^{2\mu(T-t)} \\ \beta(T) = 0 \end{cases}$$

$$\beta(t) - \beta(0) = \frac{1}{2}\sigma^2\xi^2 \int_0^t e^{2\mu(T-s)}\, ds$$

$$\Rightarrow \beta(t) = -\frac{1}{4\mu}\left(\sigma^2\xi^2\left(e^{2\mu(T-t)} - 1\right)\right).$$

$$\Rightarrow u(t,x) = \exp\left(i\xi e^{\mu(T-t)}x - \frac{1}{4\mu}\sigma^2\xi^2\left(e^{2\mu(T-t)} - 1\right)\right).$$

$$\phi_{X_T}(\xi) = u(0,x)$$

$$= \exp\left(i\xi e^{\mu T}x - \frac{1}{4\mu}\sigma^2\xi^2\left(e^{2\mu T} - 1\right)\right).$$

v). $E[X_T] = i^{-1}\phi'_{X_T}(0)$

$$= i^{-1}\left(ixe^{\mu T}\right) = xe^{\mu T}.$$

$$E[X_T^2] = i^{-2}\left(\left(ixe^{\mu T}\right)^2 + \left(-\frac{\sigma^2}{2\mu}\left(e^{2\mu T} - 1\right)\right)\right)$$

$$= x^2e^{2\mu T} + \frac{\sigma^2}{2\mu}\left(e^{2\mu T} - 1\right).$$

check.

Question 2.

i). Z_t is Gaussian and a.s. continuous, $Z_0 = 0$.

$$\mathrm{Cov}(Z_s, Z_t) = \mathbb{E}[Z_s Z_t] - \underbrace{\mathbb{E}[Z_s]\,\mathbb{E}[Z_t]}_{0}$$

$$= \mathbb{E}\Big[\Big(\sum_{j=1}^{n} c_j W_j(\alpha s)\Big)\Big(\sum_{i=1}^{n} c_i W_i(\alpha t)\Big)\Big]$$

$$= \mathbb{E}\Big[\sum_{\substack{i=1,\\ j=1}}^{n} c_i c_j W_i(\alpha t) W_j(\alpha s)\Big]$$

$$\xrightarrow{\text{ind.}} = \mathbb{E}\Big[\sum_{i=1}^{n} c_i^2 W_i(\alpha t) W_i(\alpha s)\Big]$$

$$= \sum_{i=1}^{n} c_i^2\, \mathbb{E}[W_i(\alpha t) W_i(\alpha s)]$$

$$= \sum_{i=1}^{n} c_i^2\, \mathrm{Cov}(W_i(\alpha t), W_i(\alpha s))$$

$$= \sum_{i=1}^{n} c_i^2 \cdot \alpha(t \wedge s)$$

Therefore. Z_t is a BM, iff.

$$\sum_{i=1}^{n} c_i^2 \alpha\,(s \wedge t) = s \wedge t.$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} c_i^2 \alpha = 1. \qquad \square.$$

ii). (Aim at finding a large enough b.d.d. domain, and show that its exit time is finite).

Let $S := B_{\sqrt{n}}(0)$. be the ball centered at 0 with radius $\sqrt{n}$.

Then $H \subset S$ and $\tau_H(w) \le \tau_S(w)$, for any $w \in \Omega$.

Thus $\mathbb{E}[\tau_H] \le \mathbb{E}[\tau_S] < \infty$. (b.d.dness of the ball)

Assume W_t starts with $W_0 = x$.

a) <u>Dynkins:</u>

Let $f = \|x\|^2$ and $\tau := \tau_S \wedge t$.

$$\mathbb{E}_x[f(W_\tau)] = f(x) + \mathbb{E}_x\left[\int_0^\tau \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}(W_s)\,ds\right]$$

$$= \|x\|^2 + \mathbb{E}_x\left[\int_0^\tau \frac{1}{2}\cdot 2n \; ds\right]$$

$$= \|x\|^2 + n\,\mathbb{E}_x[\tau]$$

$$f(W_\tau) \le f(W_{\tau_S}) = n$$

$$\Rightarrow \|x\|^2 + n\,\mathbb{E}_x[\tau] \le n$$

$$\|x\|^2 > 0.$$

$$\Rightarrow \quad \mathbb{E}_x[\tau] \le \frac{1}{n}(n - \|x\|^2) < 1.$$

b) Boundary value problem

Let $u(x) = \mathbb{E}_x[\tau_S]$, then $u(x)$ solves the boundary problem

$$\begin{cases} \frac{1}{2}\Delta u + 1 = 0 & \text{in } S \\ u = 0 & \text{on } \partial S \end{cases}$$

Ansatz : $u(x) = C(\|x\|^2 - n)$

$$\frac{1}{2} \cdot 2cn + 1 = 0 \quad \Rightarrow \quad C = -\frac{1}{n}.$$

$$\Rightarrow u(x) = \frac{1}{n}(n - \|x\|^2) \underset{\|x\| < 1.}{<} 1.$$

(Extra solution). (Martingale / optional sampling).

Let $M_t := W_t^2 - nt$.

Claim : M_t is a martingale.

Prf. $\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}[W_t^2 - nt \mid \mathcal{F}_s]$

$$= \mathbb{E}[(W_t - W_s + W_s)^2 - nt \mid \mathcal{F}_s]$$

$$= n(t-s) + W_s^2 - nt$$

$$= W_s^2 - ns = M_s.$$

$$M_0 = \|x\|^2 - 0 \underset{\text{optional sampling}}{=} \mathbb{E}[M_{\tau_S}]$$

$$= \mathbb{E}[W_{\tau_s}^2 - n\tau_s]$$

$$= \mathbb{E}[W_{\tau_s}^2] - n\,\mathbb{E}[\tau_s]$$

$$= n - n\,\mathbb{E}[\tau_s].$$

$$\Rightarrow \mathbb{E}[\tau_s] = \frac{n - \|x\|^2}{n} < 1. \qquad \square.$$