$\boxed{1}$. $dx = u\,dt + \sigma\,dW$.

Let $V(t,x) = \inf_u \mathbb{E}_{t,x}\left[\exp\left(\int_t^T u_s^2\,ds + X_T^2\right)\right]$.

Want: $V(0,x)$.

$$\mathcal{L}\hat{V} = u\hat{V}_x + \frac{1}{2}\sigma^2\hat{V}_{xx}$$

HJB gives

$$\begin{cases} \hat{V}_t + \inf_u \{\hat{V}u^2 + \mathcal{L}\hat{V}\} = 0 \\ \hat{V}(T,x) = e^{x^2}. \end{cases}$$

Let $\hat{V}(t,x) = \exp(A(t)x^2 + B(t))$.

$$\begin{cases} \hat{V}_t = (A_t x^2 + B_t)\hat{V} \\ \hat{V}_x = 2Ax\cdot\hat{V} \\ \hat{V}_{xx} = 2A\cdot\hat{V} + 4A^2x^2\hat{V} = 2\hat{V}(A + 2A^2x^2) \end{cases}$$

Let $f(u) = \hat{V}u^2 + \mathcal{L}u$

$$= \hat{V}u^2 + 2Axu\hat{V} + \hat{V}\sigma^2(A + 2A^2x^2)$$

$$u^* = \frac{2Ax\hat{V}}{-2\hat{V}} = -A(t)x$$

HJB now: $(A_t x^2 + B_t)\hat{V} + f(u^*) = 0$

$$\Rightarrow (A_t x^2 + B_t)\hat{V} + \hat{V}\left(A^2x^2 - 2A^2x^2 + \sigma^2(A + 2A^2x^2)\right) = 0$$

$$\Rightarrow \begin{cases} A_t + (2\sigma^2 - 1)A^2 = 0 \\ A(T) = 1. \end{cases} \qquad \begin{cases} B_t + \sigma^2 A = 0 \\ B(T) = 0 \end{cases}$$

Let $k^* = 1-2\sigma^2$, $A(t) = \frac{1}{C-k^* t}$.

$$A(T) = \frac{1}{C-k^*T} = 1 \Rightarrow C = 1+k^*T$$

$$\Rightarrow A(t) = \frac{1}{1+k^*(T-t)}.$$

$$B_t = -\frac{\sigma^2}{1+k^*(T-t)} \Rightarrow B(t) = C + \frac{\sigma^2 \ln(k^*(T-t)+1)}{k^*}.$$

$$B(T) = C = 0.$$

$$\Rightarrow B(t) = \frac{\sigma^2 \ln(k^*(T-t)+1)}{k^*}$$

$$\hat{V}(0,x) = \exp(A(0)x^2 + B(0))$$

$$= \exp\left\{ \frac{1}{1+(1-2\sigma^2)T} \cdot x^2 + \frac{\sigma^2}{1-2\sigma^2} \ln((1-2\sigma^2)T+1) \right\}$$

$$u^*(t,x) = -\frac{x}{1+(1-2\sigma^2)(T-t)}.$$

By the verification thm. $\hat{V} = V$ and u^* is optimal.

$\boxed{2}$ $V(x) = \sup_{\tau} E[e^{-\beta\tau} B_\tau^2]$.

$C := (-b, b)$.

$D := \mathbb{R} \setminus C$.

$\tau^* := \inf\{t \geq 0 : X_t \in D\}$.

Let $\hat{V}$ solve.

$$\begin{cases} \frac{1}{2}\hat{V}_{xx} - \beta\hat{V} = 0. \\ \hat{V}(-b) = \hat{V}(b) = b^2 \\ \hat{V}_x(b) = 2b. \end{cases}$$

$$\hat{V} = C\cosh(\sqrt{2\beta}\, x).$$

$$\hat{V}(b) = C\cosh(\sqrt{2\beta}\, b) = b^2$$

$$\Rightarrow \quad C = \frac{b^2}{\cosh(\sqrt{2\beta}\, b)}$$

$$\hat{V}_x(b) = C\sqrt{2\beta}\sinh(\sqrt{2\beta}\, b). = 2b.$$

$$\Rightarrow \quad b \overset{(*)}{=} \sqrt{\frac{2}{\beta}}\left(\tanh(\sqrt{2\beta}\, b)\right)^{-1}$$

$$\Rightarrow \quad \hat{V} = \begin{cases} \frac{b^2}{\cosh(\sqrt{2\beta}\, b)}\cosh(\sqrt{2\beta}\, x) & , x \in (-b, b). \\ x^2. & , x \in \mathbb{R} \setminus (-b, b). \end{cases}$$

where b is given by $(*)$

By the verification thm. $\hat{V} = V$. τ^* is an optimal strategy.